HL Paper 1

Show that the points (0, 0) and ($\sqrt{2\pi}$, $-\sqrt{2\pi}$) on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

Markscheme

attempt at implicit differentiation ***M1***

$e^{(x+y)}\left(1+\frac{dy}{dx}\right) = -\sin(xy)\left(x\frac{dy}{dx}+y\right)$ ***A1A1***

let $x = 0$, $y = 0$ ***M1***

$e^0\left(1+\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} = -1$ ***A1***

let $x = \sqrt{2\pi}$, $y = -\sqrt{2\pi}$

$e^0\left(1+\frac{dy}{dx}\right) = -\sin(-2\pi)\left(x\frac{dy}{dx}+y\right) = 0$

so $\frac{dy}{dx} = -1$ ***A1***

since both points lie on the line $y = -x$ this is a common tangent ***R1***

Note: $y = -x$ must be seen for the final ***R1***. It is not sufficient to note that the gradients are equal.

[7 marks]

Examiners report

Implicit differentiation was attempted by many candidates, some of whom obtained the correct value for the gradient of the tangent. However, very few noticed the need to go further and prove that both points were on the same line.

Let $f(x) = \sqrt{\frac{x}{1-x}}$, $0 < x < 1$.

a. Show that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ and deduce that f is an increasing function. [5]

b. Show that the curve $y = f(x)$ has one point of inflexion, and find its coordinates. [6]

c. Use the substitution $x = \sin^2\theta$ to show that $\int f(x)dx = \arcsin\sqrt{x} - \sqrt{x - x^2} + c$. [11]

Markscheme

a. **EITHER**

derivative of $\frac{x}{1-x}$ is $\frac{(1-x)-x(-1)}{(1-x)^2}$ ***M1A1***

$f'(x) = \frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}}\frac{1}{(1-x)^2}$ ***M1A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x) > 0$ (for all $0 < x < 1$) so the function is increasing ***R1***

OR

$f(x) = \frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$

$f'(x) = \frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$ ***M1A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$ ***M1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x) > 0$ (for all $0 < x < 1$) so the function is increasing ***R1***

[5 marks]

b. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$ ***M1A1***

$= -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$

$f''(x) = 0 \Rightarrow x = \frac{1}{4}$ ***M1A1***

$f''(x)$ changes sign at $x = \frac{1}{4}$ hence there is a point of inflexion ***R1***

$x = \frac{1}{4} \Rightarrow y = \frac{1}{\sqrt{3}}$ ***A1***

the coordinates are $\left(\frac{1}{4}, \frac{1}{\sqrt{3}}\right)$

[6 marks]

c. $x = \sin^2\theta \Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$ ***M1A1***

$\int\sqrt{\frac{x}{1-x}}dx = \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}2\sin\theta\cos\theta d\theta$ ***M1A1***

$= \int 2\sin^2\theta d\theta$ ***A1***

$= \int 1 - \cos 2\theta d\theta$ ***M1A1***

$= \theta - \frac{1}{2}\sin 2\theta + c$ ***A1***

$\theta = \arcsin\sqrt{x}$ ***A1***

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x-x^2}$ ***M1A1***

hence $\int\sqrt{\frac{x}{1-x}}dx = \arcsin\sqrt{x} - \sqrt{x-x^2} + c$ ***AG***

[11 marks]

Examiners report

a. Part (a) was generally well done, although few candidates made the final deduction asked for. Those that lost other marks in this part were generally due to mistakes in algebraic manipulation. In part (b) whilst many students found the second derivative and set it equal to zero, few then confirmed that it was a point of inflexion. There were several good attempts for part (c), even though there were various points throughout the question that provided stopping points for other candidates.

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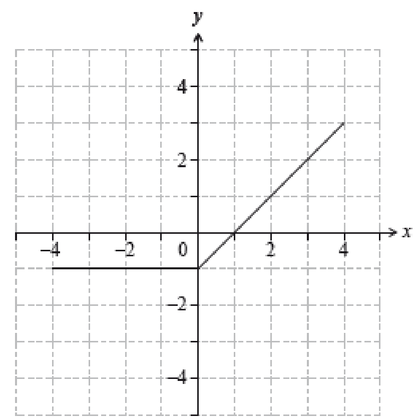
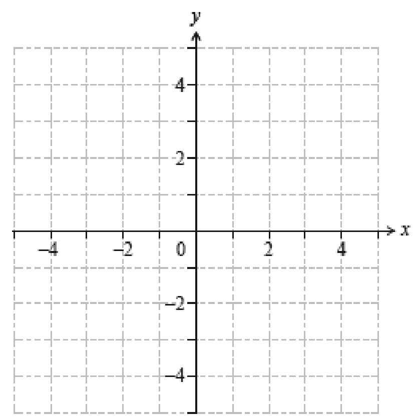
The first set of axes below shows the graph of $y = f(x)$ for $-4 \leqslant x \leqslant 4$.





Let $g(x) = \int_{-4}^{x} f(t)\mathrm{d}t$ for $-4 \leqslant x \leqslant 4$.

(a) State the value of x at which $g(x)$ is a minimum.

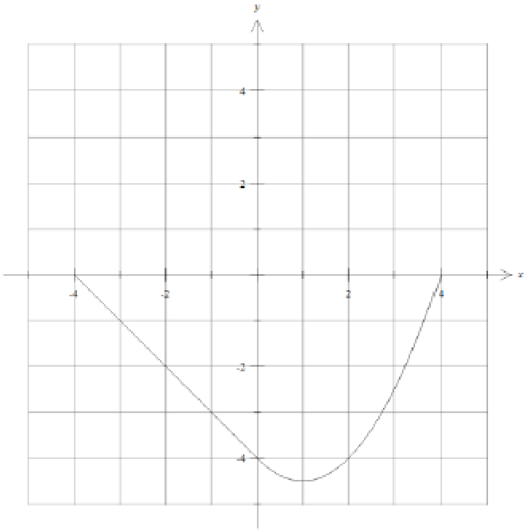
(b) On the second set of axes, sketch the graph of $y = g(x)$.

Markscheme

(a) $x = 1$ ***A1***

[1 mark]

(b) ***A1*** for point (–4, 0)
A1 for (0, – 4)
A1 for min at $x = 1$ in approximately the correct place
A1 for (4, 0)
A1 for shape including continuity at $x = 0$



[5 marks]

Total [6 marks]

Examiners report

[N/A]

The function *f* is defined by $f(x) = \mathrm{e}^{x^2 - 2x - 1.5}$.

(a) Find $f'(x)$.

(b) You are given that $y = \frac{f(x)}{x-1}$ has a local minimum at *x* = *a*, *a* > 1. Find the value of *a*.

Markscheme

(a) $\left(u = x^2 - 2x - 1.5; \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 2\right)$

$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u(2x - 2)$ ***(M1)***

$= 2(x-1)\mathrm{e}^{x^2-2x-1.5}$ ***A1***

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1) \times 2(x-1)\mathrm{e}^{x^2-2x-1.5} - 1 \times \mathrm{e}^{x^2-2x-1.5}}{(x-1)^2}$ ***M1A1***

$= \frac{2x^2-4x+1}{(x-1)^2}\mathrm{e}^{x^2-2x-1.5}$ ***(A1)***

minimum occurs when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)***

$x = 1 \pm \sqrt{\frac{1}{2}}$ $\left(\text{accept } x = \frac{4 \pm \sqrt{8}}{4}\right)$ ***A1***

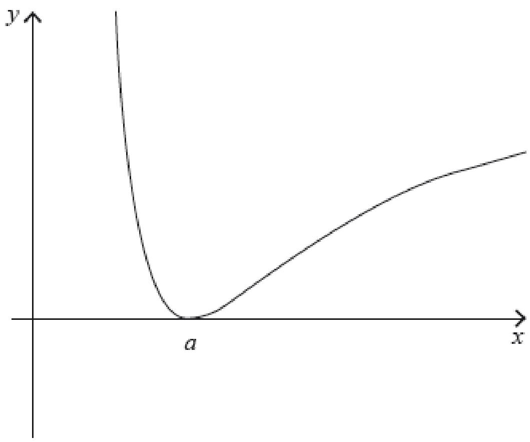
$a = 1 + \sqrt{\frac{1}{2}}$ $\left(\text{accept } a = \frac{4 + \sqrt{8}}{4}\right)$ ***R1***

[8 marks]

Examiners report

Part (a) was successfully answered by most candidates. Most candidates were able to make significant progress with part (b) but were then let down by being unable to simplify the expression or by not understanding the significance of being told that $a > 1$.

The following diagram shows the graph of $y = \frac{(\ln x)^2}{x},\ x > 0$.



The region R is enclosed by the curve, the x-axis and the line $x = e$.

Let $I_n = \int_1^{\mathrm{e}} \frac{(\ln x)^n}{x^2} \mathrm{d}x,\ n \in \mathbb{N}$.

a. Given that the curve passes through the point $(a,\ 0)$, state the value of a. [1]

b. Use the substitution $u = \ln x$ to find the area of the region R. [5]

c. (i) Find the value of I_0. [7]

(ii) Prove that $I_n = \frac{1}{\mathrm{e}} + nI_{n-1},\ n \in \mathbb{Z}^+$.

(iii) Hence find the value of I_1.

d. Find the volume of the solid formed when the region R is rotated through 2π about the x-axis. [5]

Markscheme

a. $a = 1$ ***A1***

[1 mark]

b. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***(A1)***

$\int \frac{(\ln x)^2}{x} \mathrm{d}x = \int u^2 \mathrm{d}u$ ***M1A1***

area $= \left[\frac{1}{3}u^3\right]_0^1$ or $\left[\frac{1}{3}(\ln x)^3\right]_1^{\mathrm{e}}$ ***A1***

$= \frac{1}{3}$ ***A1***

[5 marks]

c. (i) $I_0 = \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$ ***(A1)***

$= 1 - \frac{1}{\mathrm{e}}$ ***A1***

(ii) use of integration by parts ***M1***

$I_n = \left[-\frac{1}{x}(\ln x)^n\right]_1^{\mathrm{e}} + \int_1^{\mathrm{e}} \frac{n(\ln x)^{n-1}}{x^2}\mathrm{d}x$ ***A1A1***

$= -\frac{1}{\mathrm{e}} + nI_{n-1}$ ***AG***

Note: If the substitution $u = \ln x$ is used ***A1A1*** can be awarded for $I_n = [-\mathrm{e}^{-u}u^n]_0^1 + \int_0^1 n\mathrm{e}^{-u}u^{n-1}\mathrm{d}u$.

(iii) $I_1 = -\frac{1}{\mathrm{e}} + 1 \times I_0$ ***(M1)***

$= 1 - \frac{2}{\mathrm{e}}$ ***A1***

[7 marks]

d. (d) volume $= \pi \int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2}\mathrm{d}x \ (= \pi I_4)$ ***(A1)***

EITHER

$I_4 = -\frac{1}{\mathrm{e}} + 4I_3$ ***M1A1***

$= -\frac{1}{\mathrm{e}} + 4\left(-\frac{1}{\mathrm{e}} + 3I_2\right)$ ***M1***

$= -\frac{5}{\mathrm{e}} + 12I_2 = -\frac{5}{\mathrm{e}} + 12\left(-\frac{1}{\mathrm{e}} + 2I_1\right)$

OR

using parts $\int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2}\mathrm{d}x = -\frac{1}{\mathrm{e}} + 4\int_1^{\mathrm{e}} \frac{(\ln x)^3}{x^2}\mathrm{d}x$ ***M1A1***

$= -\frac{1}{\mathrm{e}} + 4\left(-\frac{1}{\mathrm{e}} + 3\int_1^{\mathrm{e}} \frac{(\ln x)^2}{x^2}\mathrm{d}x\right)$ ***M1***

THEN

$= -\frac{17}{\mathrm{e}} + 24\left(1 - \frac{2}{\mathrm{e}}\right) = 24 - \frac{65}{\mathrm{e}}$ ***A1***

volume $= \pi\left(24 - \frac{65}{\mathrm{e}}\right)$

[5 marks]

Examiners report

a. (a) and (b) were well done. Most candidates could integrate by substitution, though many did not change the limits during the substitution and, though they changed back to x at the end of their solution, under a different markscheme they might have lost marks for this in the intermediate stages.

b. (a) and (b) were well done. Most candidates could integrate by substitution, though many did not change the limits during the substitution and, though they changed back to x at the end of their solution, under a different markscheme they might have lost marks for this in the intermediate stages.

c. (c)(i) This part was well done by the candidates.

(c)(ii) This proved to be the part that was done by fewest candidates. Those who spotted that they should use integration by parts obtained the answer fairly easily.

(c)(iii) Many candidates displayed good exam technique in this question and obtained full marks without being able to do part (ii).

d. The same good exam technique was on show here as many students who failed to prove the expression in (c)(ii) were able to use it to obtain full marks in this question. A few candidates failed to remember correctly the formula for a volume of revolution.

Consider the functions $f,\ g$, defined for $x \in \mathbb{R}$, given by $f\left(x\right) = \mathrm{e}^{-x}\sin x$ and $g\left(x\right) = \mathrm{e}^{-x}\cos x$.

a.i. Find $f'\left(x\right)$. [2]

a.ii. Find $g'\left(x\right)$. [1]

b. Hence, or otherwise, find $\int\limits_0^\pi \mathrm{e}^{-x}\sin x\,\mathrm{d}x$. [4]

Markscheme

a.i. attempt at product rule ***M1***

$f'\left(x\right) = -\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x$ ***A1***

[2 marks]

a.ii. $g'\left(x\right) = -\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x$ ***A1***

[1 mark]

b. **METHOD 1**

Attempt to add $f'\left(x\right)$ and $g'\left(x\right)$ ***(M1)***

$f'\left(x\right) + g'\left(x\right) = -2\mathrm{e}^{-x}\sin x$ ***A1***

$\int\limits_0^\pi \mathrm{e}^{-x}\sin x\,\mathrm{d}x = \left[-\frac{\mathrm{e}^{-x}}{2}(\sin x + \cos x)\right]_0^\pi$ (or equivalent) ***A1***

Note: Condone absence of limits.

$= \frac{1}{2}(1 + \mathrm{e}^{-\pi})$ ***A1***

METHOD 2

$I = \int \mathrm{e}^{-x}\sin x\,\mathrm{d}x$

$= -\mathrm{e}^{-x}\cos x - \int \mathrm{e}^{-x}\cos x\,\mathrm{d}x$ **OR** $= -\mathrm{e}^{-x}\sin x + \int \mathrm{e}^{-x}\cos x\,\mathrm{d}x$ ***M1A1***

$= -\mathrm{e}^{-x}\sin x - \mathrm{e}^{-x}\cos x - \int \mathrm{e}^{-x}\sin x\,\mathrm{d}x$

$I = \frac{1}{2}\mathrm{e}^{-x}\left(\sin x + \cos x\right)$ ***A1***

$\int_0^\pi \mathrm{e}^{-x}\sin x\,\mathrm{d}x = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$ ***A1***

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

A drinking glass is modelled by rotating the graph of $y = \mathrm{e}^x$ about the y-axis, for $1 \leqslant y \leqslant 5$. Find the volume of the glass.

Markscheme

$y = \mathrm{e}^x \Rightarrow x = \ln y$

volume $= \pi \int_1^5 (\ln y)^2 \mathrm{d}y$ ***(M1)A1***

using integration by parts ***(M1)***

$\pi \int_1^5 (\ln y)^2 \mathrm{d}y = \pi \left[y(\ln y)^2 \right]_1^5 - 2 \int_1^5 \ln y \mathrm{d}y$ ***A1A1***

$= \pi \left[y(\ln y)^2 - 2y \ln y + 2y \right]_1^5$ ***A1A1***

Note: Award ***A1*** marks if π is present in at least one of the above lines.

$\Rightarrow \pi \int_1^5 (\ln y)^2 \mathrm{d}y = \pi\, 5(\ln 5)^2 - 10 \ln 5 + 8$ ***A1***

[8 marks]

Examiners report

Only the best candidates were able to make significant progress with this question. Quite a few did not consider rotation about the y-axis. Others wrote the correct expression, but seemed daunted by needing to integrate by parts twice.

A curve is defined by $xy = y^2 + 4$.

a. Show that there is no point where the tangent to the curve is horizontal. [4]

b. Find the coordinates of the points where the tangent to the curve is vertical. [4]

Markscheme

a. $x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 2y \frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1***

a horizontal tangent occurs if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so $y = 0$ ***M1***

we can see from the equation of the curve that this solution is not possible $(0 = 4)$ and so there is not a horizontal tangent ***R1***

[4 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2y - x}$ or equivalent with $\frac{\mathrm{d}x}{\mathrm{d}y}$

the tangent is vertical when $2y = x$ ***M1***

substitute into the equation to give $2y^2 = y^2 + 4$ ***M1***

$y = \pm 2$ ***A1***

coordinates are $(4,\ 2),\ (-4,\ -2)$ ***A1***

[4 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the function defined by $f(x) = x\sqrt{1-x^2}$ on the domain $-1 \le x \le 1$.

a. Show that f is an odd function. [2]

b. Find $f'(x)$. [3]

c. Hence find the x-coordinates of any local maximum or minimum points. [3]

d. Find the range of f. [3]

e. Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

f. Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \ge 0$. [4]

g. Show that $\int_{-1}^{1} \left|x\sqrt{1-x^2}\right| \mathrm{d}x > \left|\int_{-1}^{1} x\sqrt{1-x^2}\mathrm{d}x\right|$. [2]

Markscheme

a. $f(-x) = (-x)\sqrt{1-(-x)^2}$ ***M1***

$= -x\sqrt{1-x^2}$

$= -f(x)$ ***R1***

hence f is odd ***AG***

[2 marks]

b. $f'(x) = x \bullet \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \bullet -2x + (1-x^2)^{\frac{1}{2}}$ ***M1A1A1***

[3 marks]

c. $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$ $\left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ ***A1***

Note: This may be seen in part (b).

Note: Do not allow FT from part (b).

$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$ ***M1***

$x = \pm\frac{1}{\sqrt{2}}$ ***A1***

[3 marks]

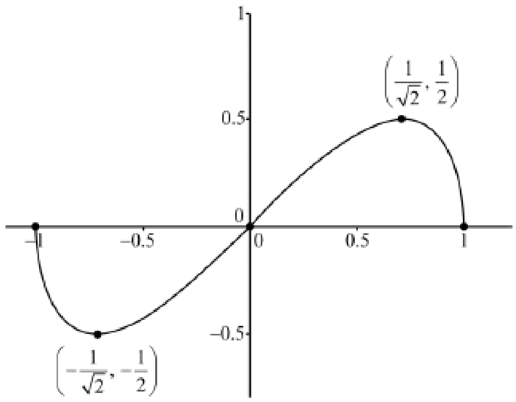
d. y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ ***M1A1***

so range of $f(x)$ is $\left[-\frac{1}{2},\ \frac{1}{2}\right]$ ***A1***

Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

e.



Shape: The graph of an odd function, on the given domain, s-shaped,

where the max(min) is the right(left) of $0.5\ (-0.5)$ ***A1***

x-intercepts ***A1***

turning points ***A1***

[3 marks]

f. $\text{area} = \int_0^1 x\sqrt{1-x^2}\mathrm{d}x$ ***(M1)***

attempt at “backwards chain rule” or substitution ***M1***

$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\mathrm{d}x$

Note: Condone absence of limits for first two marks.

$= \left[\frac{2}{3}(1-x^2)^{\frac{3}{2}} \bullet -\frac{1}{2}\right]_0^1$ ***A1***

$= \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ ***A1***

[4 marks]

g. $\int_{-1}^1 \left|x\sqrt{1-x^2}\right| \mathrm{d}x > 0$ ***R1***

$\left|\int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x\right| = 0$ ***R1***

so $\int_{-1}^1 \left|x\sqrt{1-x^2}\right| \mathrm{d}x > \left|\int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x\right|$ ***AG***

[2 marks]

Total [20 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]
g. [N/A]

Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. [7]

c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. [5]

Justify whether any such point is a maximum or a minimum.

d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

e. Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = (\mathrm{e}^{3x} + 3x\mathrm{e}^{3x})$ ***M1A1***

[2 marks]

b. let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as $\mathrm{LHS} = \mathrm{RHS}$, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true $(\forall n \in \mathbb{Z}^+)$ ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

c. $\mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3},\ -\frac{1}{3\mathrm{e}}\right)$ ***A1***

EITHER

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$

when $x = -\frac{1}{3}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-ve$ 0 $+ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

d. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$ ***A1***

$2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3},\ -\frac{2}{3\mathrm{e}^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	$-ve$ 0 $+ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

Note: Allow 3^{rd} derivative is not zero at $-\frac{2}{3}$

[5 marks]

e.

(general shape including asymptote and through origin) ***A1***

showing minimum and point of inflection ***A1***

Note: Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

Examiners report

a. Well done.

b. The logic of an induction proof was not known well enough. Many candidates used what they had to prove rather than differentiating what they had assumed. They did not have enough experience in doing Induction proofs.

c. Good, some forgot to test for min/max, some forgot to give the y value.

d. Again quite good, some forgot to check for change in curvature and some forgot the y value.

e. Some accurate sketches, some had all the information from earlier parts but could not apply it. The asymptote was often missed.

Using integration by parts find $\int x \sin x \mathrm{d}x$.

Markscheme

attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x \mathrm{d}x = x(-\cos x) + \int \cos x \mathrm{d}x$ ***(A1)(A1)***

$= -x \cos x + \sin x + c$ ***A1***

Note: Only award final ***A1*** if $+c$ is seen.

[4 marks]

Examiners report

[N/A]

By using the substitution $u = \mathrm{e}^x + 3$, find $\int \frac{\mathrm{e}^x}{\mathrm{e}^{2x}+6\mathrm{e}^x+13}\mathrm{d}x$.

Markscheme

$\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ ***(A1)***

EITHER

integral is $\int \frac{\mathrm{e}^x}{(\mathrm{e}^x+3)^2+2^2}\mathrm{d}x$ ***M1A1***

$= \frac{1}{u^2+2^2}\mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

Note: $\mathrm{d}u$ needed for final ***A1***

OR

$\mathrm{e}^x = u - 3$

integral is $\int \frac{1}{(u-3)^2+6(u-3)+13}\mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

$= \int \frac{1}{u^2+2^2}\mathrm{d}u$ ***M1A1***

Note: In both solutions the two method marks are independent.

THEN

$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)(+c)$ ***(A1)***

$= \frac{1}{2}\arctan\left(\frac{e^x+3}{2}\right)(+c)$ ***A1***

Total [7 marks]

Examiners report

Many good complete answers. Some did not realise it was arctan. Some had poor understanding of the method.

Consider the function defined by $f(x) = x^3 - 3x^2 + 4$.

a. Determine the values of x for which $f(x)$ is a decreasing function. [4]

b. There is a point of inflexion, P, on the curve $y = f(x)$. [3]

Find the coordinates of P.

Markscheme

a. attempt to differentiate $f(x) = x^3 - 3x^2 + 4$ ***M1***

$f'(x) = 3x^2 - 6x$ ***A1***

$= 3x(x-2)$

(Critical values occur at) $x = 0,\ x = 2$ ***(A1)***

so f decreasing on $x \in]0,\ 2[\quad (\text{or } 0 < x < 2)$ ***A1***

[4 marks]

b. $f''(x) = 6x - 6$ ***(A1)***

setting $f''(x) = 0$ ***M1***

$\Rightarrow x = 1$

coordinate is $(1,\ 2)$ ***A1***

[3 marks]

Total [7 marks]

Examiners report

a. [N/A]
b. [N/A]

---

a. Show that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. [1]

b. Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

Markscheme

a. $\sin\left(\theta+\frac{\pi}{2}\right)=\sin\theta\cos\frac{\pi}{2}+\cos\theta\sin\frac{\pi}{2}$ ***M1***

$=\cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

b. consider $n=1,\ f'(x)=a\cos(ax)$ ***M1***

since $\sin\left(ax+\frac{\pi}{2}\right)=\cos ax$ then the proposition is true for $n=1$ ***R1***

assume that the proposition is true for $n=k$ so $f^{(k)}(x)=a^k\sin\left(ax+\frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x)=\frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x}\quad\left(=a\left(a^k\cos\left(ax+\frac{k\pi}{2}\right)\right)\right)$ ***M1***

$=a^{k+1}\sin\left(ax+\frac{k\pi}{2}+\frac{\pi}{2}\right)$ (using part (a)) ***A1***

$=a^{k+1}\sin\left(ax+\frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n=k$ then we have shown that the proposition is true for $n=k+1$. Since we have shown that the proposition is true for $n=1$ then the proposition is true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior ***M*** and ***R*** marks have been awarded.

[7 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

---

a. Find the value of the integral $\int_0^{\sqrt{2}}\sqrt{4-x^2}\mathrm{d}x$. [7]

b. Find the value of the integral $\int_0^{0.5}\arcsin x\mathrm{d}x$. [5]

c. Using the substitution $t=\tan\theta$, find the value of the integral [7]

$$\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}\theta}{3\cos^2\theta+\sin^2\theta}.$$

Markscheme

a. let $x=2\sin\theta$ ***M1***

$\mathrm{d}x=2\cos\theta\mathrm{d}\theta$ ***A1***

$I=\int_0^{\frac{\pi}{4}}2\cos\theta\times 2\cos\theta\mathrm{d}\theta\quad\left(=4\int_0^{\frac{\pi}{4}}\cos^2\theta\mathrm{d}\theta\right)$ ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for expression.

$= 2\int_0^{\frac{\pi}{4}} (1 + \cos 2\theta)\mathrm{d}\theta$ ***A1***

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ ***A1***

$= 1 + \frac{\pi}{2}$ ***A1***

[7 marks]

b. $I = [x \arcsin x]_0^{0.5} - \int_0^{0.5} x \times \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ ***M1A1A1***

$= [x \arcsin x]_0^{0.5} + \left[\sqrt{1 - x^2}\right]_0^{0.5}$ ***A1***

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ ***A1***

[5 marks]

c. $\mathrm{d}t = \sec^2\theta \mathrm{d}\theta$, $\left[0, \frac{\pi}{4}\right] \to [0,\ 1]$ ***A1(A1)***

$I = \int_0^1 \frac{\frac{\mathrm{d}t}{(1+t^2)}}{\frac{3}{(1+t^2)} + \frac{t^2}{(1+t^2)}}$ ***M1(A1)***

$= \int_0^1 \frac{\mathrm{d}t}{3+t^2}$ ***A1***

$= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$ ***A1***

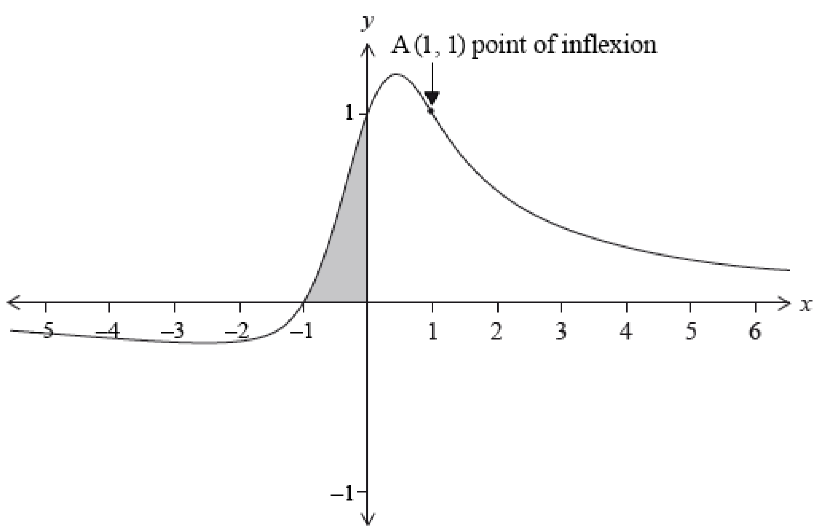
$= \frac{\pi}{6\sqrt{3}}$ ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



The point (1, 1) is a point of inflexion. There are two other points of inflexion.

a. Find $f'(x)$. [2]

b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero. [1]

c. Find $f''(x)$ expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. [3]

d. Find the x-coordinates of the other two points of inflexion. [4]

e. Find the area of the shaded region. Express your answer in the form $\frac{\pi}{a} - \ln\sqrt{b}$, where a and b are integers. [6]

Markscheme

a. (a) $f'(x) = \frac{(x^2+1)-2x(x+1)}{(x^2+1)^2} \left(= \frac{-x^2-2x+1}{(x^2+1)^2}\right)$ ***M1A1***

[2 marks]

b. $\frac{-x^2-2x+1}{(x^2+1)^2} = 0$

$x = -1 \pm \sqrt{2}$ ***A1***

[1 mark]

c. $f''(x) = \frac{(-2x-2)(x^2+1)^2 - 2(2x)(x^2+1)(-x^2-2x+1)}{(x^2+1)^4}$ ***A1A1***

Note: Award ***A1*** for $(-2x-2)(x^2+1)^2$ or equivalent.

Note: Award ***A1*** for $-2(2x)(x^2+1)(-x^2-2x+1)$ or equivalent.

$= \frac{(-2x-2)(x^2+1)-4x(-x^2-2x+1)}{(x^2+1)^3}$

$= \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$ ***A1***

$\left(= \frac{2(x^3+3x^2-3x-1)}{(x^2+1)^3}\right)$

[3 marks]

d. recognition that $(x-1)$ is a factor ***(R1)***

$(x-1)(x^2+bx+c) = (x^3+3x^2-3x-1)$ ***M1***

$\Rightarrow x^2+4x+1 = 0$ ***A1***

$x = -2 \pm \sqrt{3}$ ***A1***

Note: Allow long division / synthetic division.

[4 marks]

e. $\int_{-1}^{0} \frac{x+1}{x^2+1} \mathrm{d}x$ ***M1***

$\int \frac{x+1}{x^2+1} \mathrm{d}x = \int \frac{x}{x^2+1} \mathrm{d}x + \int \frac{1}{x^2+1} \mathrm{d}x$ ***M1***

$= \frac{1}{2}\ln(x^2+1) + \arctan(x)$ ***A1A1***

$= \left[\frac{1}{2}\ln(x^2+1) + \arctan(x)\right]_{-1}^{0} = \frac{1}{2}\ln 1 + \arctan 0 - \frac{1}{2}\ln 2 - \arctan(-1)$ ***M1***

$= \frac{\pi}{4} - \ln\sqrt{2}$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

e. [N/A]

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

a. Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta,\ n \in \mathbb{Z}^+$. [2]

b. Expand $\left(z + z^{-1}\right)^4$. [1]

c. Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where $p,\ q$ and r are constants to be determined. [4]

d. Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. [3]

e. Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \mathrm{d}\theta$. [3]

f. S is rotated through 2π radians about the x-axis. Find the value of the volume generated. [4]

g. (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$. [3]

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta$ in terms of k.

Markscheme

a. $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***

$= 2\cos n\theta$ ***AG***

[2 marks]

b. (b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

c. **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

d. $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6 = \left(z^6+\frac{1}{z^6}\right) + 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ ***AG***

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

e. $\int_0^{\frac{\pi}{2}} \cos^6\theta \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) \mathrm{d}\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

f. $\mathrm{V} = \pi\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \mathrm{d}x$ ***M1***

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x \mathrm{d}x - \pi\int_0^{\frac{\pi}{2}} \cos^6 x \mathrm{d}x$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x \mathrm{d}x = \frac{3\pi}{16}$ ***A1***

$\mathrm{V} = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

Note: Follow through from an incorrect *r* in (c) provided the final answer is positive.

g. (i) constant term $= \begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2} \left(\text{or} \frac{\begin{pmatrix} 2k \\ k \end{pmatrix}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Examiners report

a. Part a) has appeared several times before, though with it again being a ‘show that’ question, some candidates still need to be more aware of the need to show every step in their working, including the result that $\sin(-n\theta) = -\sin(n\theta)$.

b. Part b) was usually answered correctly.

c. Part c) was again often answered correctly, though some candidates often less successfully utilised a trig-only approach rather than taking note of part b).

d. Part d) was a good source of marks for those who kept with the spirit of using complex numbers for this type of question. Some limited attempts at trig-only solutions were seen, and correct solutions using this approach were extremely rare.

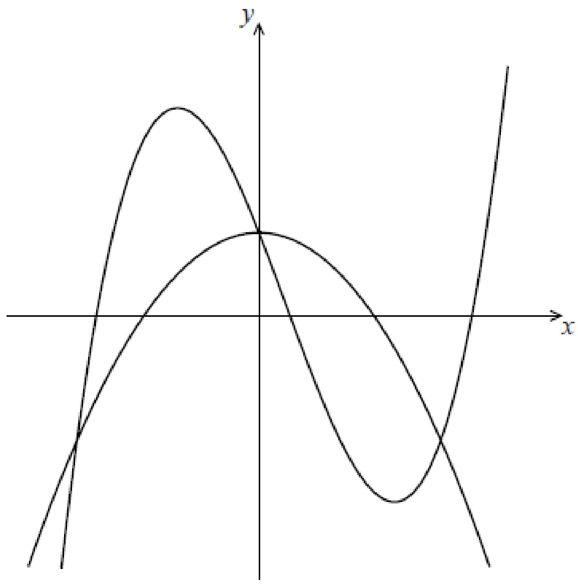
e. Part e) was well answered, though numerical slips were often common. A small number integrated $\sin n\theta$ as $n\cos n\theta$.

A large number of candidates did not realise the help that part e) inevitably provided for part f). Some correctly expressed the volume as $\pi\int\cos^4 x\mathrm{d}x - \pi\int\cos^6 x\mathrm{d}x$ and thus gained the first 2 marks but were able to progress no further. Only a small number of able candidates were able to obtain the correct answer of $\frac{\pi^2}{32}$.

[N/A]

g. Part g) proved to be a challenge for the vast majority, though it was pleasing to see some of the highest scoring candidates gain all 3 marks.

The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - bx + 2,\ b > 0$, intersect and create two closed regions. Show that these two regions have equal areas.



Markscheme

to find the points of intersection of the two curves

$-x^2 + 2 = x^3 - x^2 - bx + 2$ ***M1***

$x^3 - bx = x(x^2 - b) = 0$

$\Rightarrow x = 0;\ x = \pm\sqrt{b}$ ***A1A1***

$A_1 = \int_{-\sqrt{b}}^{0} \left[(x^3 - x^2 - bx + 2) - (-x^2 + 2)\right] \mathrm{d}x \left(= \int_{-\sqrt{b}}^{0} (x^3 - bx)\mathrm{d}x\right)$ ***M1***

$= \left[\frac{x^4}{4} - \frac{bx^2}{2}\right]_{-\sqrt{b}}^{0}$

$= -\left(\frac{(-\sqrt{b})^4}{4} - \frac{b(-\sqrt{b})^2}{2}\right) = -\frac{b^2}{4} + \frac{b^2}{2} = \frac{b^2}{4}$ ***A1***

$A_2 = \int_{0}^{\sqrt{b}} \left[(-x^2 + 2) - (x^3 - x^2 - bx + 2)\right] \mathrm{d}x$ ***M1***

$= \int_{0}^{\sqrt{b}} (-x^3 + bx)\mathrm{d}x$

$= \left[-\frac{x^4}{4} + \frac{bx^2}{2}\right]_{0}^{\sqrt{b}} = \frac{b^2}{4}$ ***A1***

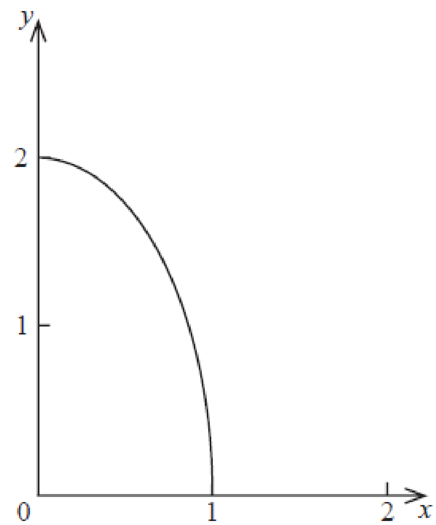
therefore $A_1 = A_2 = \frac{b^2}{4}$ ***AG***

[7 marks]

Examiners report

Most candidates knew how to tackle this question. The most common error was in giving $+b$ and $-b$ as the x-coordinates of the point of intersection.

Consider the part of the curve $4x^2 + y^2 = 4$ shown in the diagram below.



(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y .

(b) Find the gradient of the tangent at the point $\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

(c) A bowl is formed by rotating this curve through 2π radians about the x-axis.

Calculate the volume of this bowl.

Markscheme

(a) $8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1A1***

Note: Award ***M1A0*** for $8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{y}$ ***A1***

(b) -4 ***A1***

(c) $V = \int \pi y^2 \mathrm{d}x$ or equivalent ***M1***

$V = \pi \int_0^1 (4 - 4x^2)\mathrm{d}x$ ***A1***

$= \pi\left[4x - \frac{4}{3}x^3\right]_0^1$ ***A1***

$= \frac{8\pi}{3}$ ***A1***

Note: If it is correct except for the omission of π , award 2 marks.

[8 marks]

Examiners report

The first part of this question was done well by many, the only concern being the number that did not simplify the result from $-\frac{8x}{2y}$. There were many variations on the formula for the volume in part c), the most common error being a multiple of 2π rather than π. On the whole this question was done well by many.

The curve C with equation $y = f(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2),\ y > 1,$$

and $y = \mathrm{e}$ when $x = 2$.

a. Find the equation of the tangent to C at the point (2, e). [3]

b. Find $f(x)$. [11]

c. Determine the largest possible domain of f. [6]

d. Show that the equation $f(x) = f'(x)$ has no solution. [4]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}}{\ln \mathrm{e}}(2+2) = 4\mathrm{e}$ ***A1***

at (2, e) the tangent line is $y - \mathrm{e} = 4\mathrm{e}(x-2)$ ***M1***

hence $y = 4\mathrm{e}x - 7\mathrm{e}$ ***A1***

[3 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}\mathrm{d}y = (x+2)\mathrm{d}x$ ***M1***

$\int \frac{\ln y}{y}\mathrm{d}y = \int (x+2)\mathrm{d}x$

using substitution $u = \ln y;\ \mathrm{d}u = \frac{1}{y}\mathrm{d}y$ ***(M1)(A1)***

$\Rightarrow \int \frac{\ln y}{y}\mathrm{d}y = \int u\mathrm{d}u = \frac{1}{2}u^2$ ***(A1)***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ ***A1A1***

at (2, e), $\frac{(\ln \mathrm{e})^2}{2} = 6 + c$ ***M1***

$\Rightarrow c = -\frac{11}{2}$ ***A1***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2+4x-11} \Rightarrow y = \mathrm{e}^{\pm\sqrt{x^2+4x-11}}$ ***M1A1***

since y > 1, $f(x) = \mathrm{e}^{\sqrt{x^2+4x-11}}$ ***R1***

Note: ***M1*** for attempt to make *y* the subject.

[11 marks]

c. **EITHER**

$x^2 + 4x - 11 > 0$ ***A1***

using the quadratic formula ***M1***

critical values are $\frac{-4\pm\sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$ ***A1***

using a sign diagram or algebraic solution ***M1***

$x < -2-\sqrt{15};\ x > -2+\sqrt{15}$ ***A1A1***

OR

$x^2+4x-11>0$ ***A1***

by methods of completing the square ***M1***

$(x+2)^2 > 15$ ***A1***

$\Rightarrow x+2 < -\sqrt{15}$ or $x+2 > \sqrt{15}$ ***(M1)***

$x < -2-\sqrt{15};\ x > -2+\sqrt{15}$ ***A1A1***

[6 marks]

d. $f(x) = f'(x) \Rightarrow f(x) = \frac{f(x)}{\ln f(x)}(x+2)$ ***M1***

$\Rightarrow \ln(f(x)) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$ ***A1***

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$ ***A1***

$\Rightarrow 4 = -11,\ \text{hence } f(x) \ne f'(x)$ ***R1AG***

[4 marks]

Examiners report

a. Nearly always correctly answered.

b. Most candidates separated the variables and attempted the integrals. Very few candidates made use of the condition $y > 1$, so losing 2 marks.

c. Part (c) was often well answered, sometimes with follow through.

d. Only the best candidates were successful on part (d).

Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$.

Markscheme

METHOD 1

$\text{area} = \int\limits_0^{\sqrt{3}} \arctan x \mathrm{d}x$ ***A1***

attempting to integrate by parts ***M1***

$= [x \arctan x]_0^{\sqrt{3}} - \int\limits_0^{\sqrt{3}} x \frac{1}{1+x^2} \mathrm{d}x$ ***A1A1***

$= [x \arctan x]_0^{\sqrt{3}} - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^{\sqrt{3}}$ ***A1***

Note: Award A1 even if limits are absent.

$= \frac{\pi}{\sqrt{3}} - \frac{1}{2}\ln 4$ ***A1***

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

METHOD 2

$\text{area} = \frac{\pi\sqrt{3}}{3} - \int\limits_0^{\frac{\pi}{3}} \tan y \mathrm{d}y$ ***M1A1A1***

$= \frac{\pi\sqrt{3}}{3} + [\ln|\cos y|]_0^{\frac{\pi}{3}}$ ***M1A1***

$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} \quad \left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$ ***A1***

[6 marks]

Examiners report

Many candidates were able to write down the correct expression for the required area, although in some cases with incorrect integration limits. However, very few managed to achieve any further marks due to a number of misconceptions, in particular $\arctan x = \cot x = \frac{\cos x}{\sin x}$. Candidates who realised they should use integration by parts were in general very successful in answering this question. It was pleasing to see a few alternative correct approaches to this question.

A curve has equation $3x - 2y^2 e^{x-1} = 2$.

a. Find an expression for $\frac{dy}{dx}$ in terms of x and y. [5]

b. Find the equations of the tangents to this curve at the points where the curve intersects the line $x = 1$. [4]

Markscheme

a. attempt to differentiate implicitly ***M1***

$3 - \left(4y\frac{dy}{dx} + 2y^2\right)e^{x-1} = 0$ ***A1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$\frac{dy}{dx} = \frac{3 \bullet e^{1-x} - 2y^2}{4y}$ ***A1***

Note: This final answer may be expressed in a number of different ways.

[5 marks]

b. $3 - 2y^2 = 2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$ ***A1***

$\frac{dy}{dx} = \frac{3 - 2 \bullet \frac{1}{2}}{\pm 4\sqrt{\frac{1}{2}}} = \pm\frac{\sqrt{2}}{2}$ ***M1***

at $\left(1, \sqrt{\frac{1}{2}}\right)$ the tangent is $y - \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}(x - 1)$ and ***A1***

at $\left(1, -\sqrt{\frac{1}{2}}\right)$ the tangent is $y + \sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}(x - 1)$ ***A1***

Note: These equations simplify to $y = \pm\frac{\sqrt{2}}{2}x$.

Note: Award ***A0M1A1A0*** if just the positive value of y is considered and just one tangent is found.

[4 marks]

Examiners report

a. [N/A]
b. [N/A]

A curve has equation $\arctan x^2 + \arctan y^2 = \frac{\pi}{4}$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(b) Find the gradient of the curve at the point where $x = \frac{1}{\sqrt{2}}$ and $y < 0$.

Markscheme

(a) **METHOD 1**

$\frac{2x}{1+x^4} + \frac{2y}{1+y^4}\frac{dy}{dx} = 0$ ***M1A1A1***

Note: Award ***M1*** for implicit differentiation, ***A1*** for LHS and ***A1*** for RHS.

$\frac{dy}{dx} = -\frac{x(1+y^4)}{y(1+x^4)}$ ***A1***

METHOD 2

$y^2 = \tan\left(\frac{\pi}{4} - \arctan x^2\right)$

$= \frac{\tan\frac{\pi}{4} - \tan(\arctan x^2)}{1+\left(\tan\frac{\pi}{4}\right)(\tan(\arctan x^2))}$ ***(M1)***

$= \frac{1-x^2}{1+x^2}$ ***A1***

$2y\frac{dy}{dx} = \frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}$ ***M1***

$2y\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$

$\frac{dy}{dx} = -\frac{2x}{y(1+x^2)^2}$ ***A1***

$\left(= \frac{2x\sqrt{1+x^2}}{\sqrt{1-x^2}(1+x^2)^2}\right)$

[4 marks]

(b) $y^2 = \tan\left(\frac{\pi}{4} - \arctan\frac{1}{2}\right)$ ***(M1)***

$= \frac{\tan\frac{\pi}{4} - \tan\left(\arctan\frac{1}{2}\right)}{1+\left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan\frac{1}{2}\right)\right)}$ ***(M1)***

Note: The two ***M1***s may be awarded for working in part (a).

$= \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$ ***A1***

$y = -\frac{1}{\sqrt{3}}$ ***A1***

substitution into $\frac{dy}{dx}$

$= \frac{4\sqrt{6}}{9}$ ***A1***

Note: Accept $\frac{8\sqrt{3}}{9\sqrt{2}}$ *etc.*

[5 marks]

Total [9 marks]

Examiners report

[N/A]

Let $x^3y = a\sin nx$. Using implicit differentiation, show that

$$x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x^2\frac{\mathrm{d}y}{\mathrm{d}x} + (n^2x^2 + 6)xy = 0$$

.

Markscheme

$x^3y = a\sin nx$
attempt to differentiate implicitly ***M1***
$\Rightarrow 3x^2y + x^3\frac{\mathrm{d}y}{\mathrm{d}x} = an\cos nx$ ***A2***
Note: Award ***A1*** for two out of three correct, ***A0*** otherwise.

$\Rightarrow 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -an^2\sin nx$ ***A2***
Note: Award ***A1*** for three or four out of five correct, ***A0*** otherwise.

$\Rightarrow 6xy + 6x^2\frac{\mathrm{d}y}{\mathrm{d}x} + x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -an^2\sin nx$
$\Rightarrow x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 6xy + n^2x^3y = 0$ ***A1***
$\Rightarrow x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x^2\frac{\mathrm{d}y}{\mathrm{d}x} + (n^2x^2 + 6)xy = 0$ ***AG***
[6 marks]

Examiners report

Candidates who are comfortable using implicit differentiation found this to be a fairly straightforward question and were able to answer it in just a few lines. Many candidates, however, were unable to differentiate x^3y with respect to *x* and were therefore unable to proceed. Candidates whose first step was to write $y = \frac{a\sin nx}{x^3}$ were given no credit since the question required the use of implicit differentiation.

Let $y = \mathrm{e}^x\sin x$.

Consider the function f defined by $f(x) = \mathrm{e}^x\sin x,\ 0 \leqslant x \leqslant \pi$.

The curvature at any point $(x,\ y)$ on a graph is defined as $\kappa = \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}$.

a. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\mathrm{e}^x \cos x$. [2]

c. Show that the function f has a local maximum value when $x = \frac{3\pi}{4}$. [2]

d. Find the x-coordinate of the point of inflexion of the graph of f. [2]

e. Sketch the graph of f, clearly indicating the position of the local maximum point, the point of inflexion and the axes intercepts. [3]

f. Find the area of the region enclosed by the graph of f and the x-axis. [6]

The curvature at any point $(x,\ y)$ on a graph is defined as $\kappa = \frac{\left|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|}{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}$.

g. Find the value of the curvature of the graph of f at the local maximum point. [3]

h. Find the value κ for $x = \frac{\pi}{2}$ and comment on its meaning with respect to the shape of the graph. [2]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x \ \left(= \mathrm{e}^x(\sin x + \cos x)\right)$ ***M1A1***

[2 marks]

b. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^x(\sin x + \cos x) + \mathrm{e}^x(\cos x - \sin x)$ ***M1A1***

$= 2\mathrm{e}^x \cos x$ ***AG***

[2 marks]

c. $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\frac{3\pi}{4}}\left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) = 0$ ***R1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\mathrm{e}^{\frac{3\pi}{4}}\cos\frac{3\pi}{4} < 0$ ***R1***

hence maximum at $x = \frac{3\pi}{4}$ ***AG***

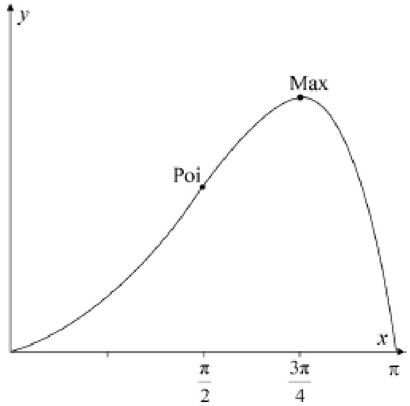
[2 marks]

d. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow 2\mathrm{e}^x \cos x = 0$ ***M1***

$\Rightarrow x = \frac{\pi}{2}$ ***A1***

Note: Award ***M1A0*** if extra zeros are seen.

[2 marks]

e. 

correct shape and correct domain ***A1***

max at $x=\frac{3\pi}{4}$, point of inflexion at $x=\frac{\pi}{2}$ ***A1***

zeros at $x=0$ and $x=\pi$ ***A1***

Note: Penalize incorrect domain with first ***A*** mark; allow ***FT*** from (d) on extra points of inflexion.

[3 marks]

f. **EITHER**

$\int_0^x \mathrm{e}^x \sin x \mathrm{d}x = [\mathrm{e}^x \sin x]_0^\pi - \int_0^\pi \mathrm{e}^x \cos x \mathrm{d}x$ ***M1A1***

$\int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x = [\mathrm{e}^x \sin x]_0^\pi - \left([\mathrm{e}^x \cos x]_0^x + \int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x\right)$ ***A1***

OR

$\int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x = [-\mathrm{e}^x \cos x]_0^\pi + \int_0^\pi \mathrm{e}^x \cos x \mathrm{d}x$ ***M1A1***

$\int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x = [-\mathrm{e}^x \cos x]_0^\pi + \left([\mathrm{e}^x \sin x]_0^\pi - \int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x\right)$ ***A1***

THEN

$\int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x = \frac{1}{2}\left([\mathrm{e}^x \sin x]_0^x - [\mathrm{e}^x \cos x]_0^x\right)$ ***M1A1***

$\int_0^\pi \mathrm{e}^x \sin x \mathrm{d}x = \frac{1}{2}(\mathrm{e}^x + 1)$ ***A1***

[6 marks]

g. $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(A1)***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}} \cos \frac{3\pi}{4} = -\sqrt{2}e^{\frac{3\pi}{4}}$ ***(A1)***

$\kappa = \frac{\left|-\sqrt{2}\mathrm{e}^{\frac{3\pi}{4}}\right|}{1} = \sqrt{2}\mathrm{e}^{\frac{3\pi}{4}}$ ***A1***

[3 marks]

h. $\kappa = 0$ ***A1***

the graph is approximated by a straight line ***R1***

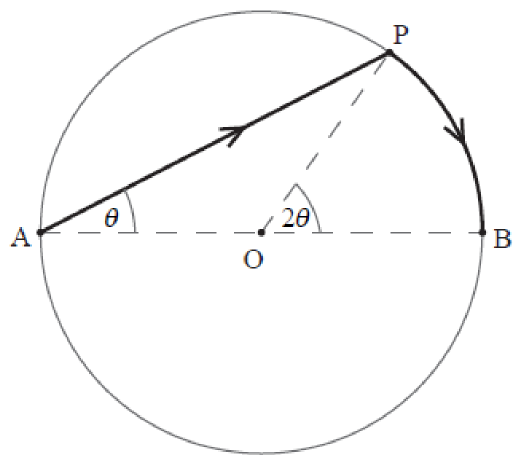
[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

f. [N/A]
g. [N/A]
h. [N/A]

The diagram below shows a circular lake with centre O, diameter AB and radius 2 km.



Jorg needs to get from A to B as quickly as possible. He considers rowing to point P and then walking to point B. He can row at $3\ \mathrm{km\,h^{-1}}$ and walk at $6\ \mathrm{km\,h^{-1}}$. Let $\mathrm{P\hat{A}B} = \theta$ radians, and t be the time in hours taken by Jorg to travel from A to B.

a. Show that $t = \frac{2}{3}(2\cos\theta + \theta)$. [3]

b. Find the value of θ for which $\frac{\mathrm{d}t}{\mathrm{d}\theta} = 0$. [2]

c. What route should Jorg take to travel from A to B in the least amount of time? [3]

Give reasons for your answer.

Markscheme

a. angle APB is a right angle

$\Rightarrow \cos\theta = \frac{\mathrm{AP}}{4} \Rightarrow \mathrm{AP} = 4\cos\theta$ ***A1***

Note: Allow correct use of cosine rule.

arc $\mathrm{PB} = 2 \times 2\theta = 4\theta$ ***A1***

$t = \frac{\mathrm{AP}}{3} + \frac{\mathrm{PB}}{6}$ ***M1***

Note: Allow use of their AP and their PB for the ***M1***.

$\Rightarrow t = \frac{4\cos\theta}{3} + \frac{4\theta}{6} = \frac{4\cos\theta}{3} + \frac{2\theta}{3} = \frac{2}{3}(2\cos\theta + \theta)$ ***AG***

[3 marks]

b. $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{2}{3}(-2\sin\theta + 1)$ ***A1***

$\frac{2}{3}(-2\sin\theta + 1) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ (or 30 degrees) ***A1***

[2 marks]

c. $\frac{d^2t}{d\theta^2} = -\frac{4}{3}\cos\theta < 0 \quad \left(\text{at } \theta = \frac{\pi}{6}\right)$ ***M1***

$\Rightarrow t$ is maximized at $\theta = \frac{\pi}{6}$ ***R1***

time needed to walk along arc AB is $\frac{2\pi}{6}$ $(\approx 1 \text{ hour})$

time needed to row from A to B is $\frac{4}{3}$ $(\approx 1.33 \text{ hour})$

hence, time is minimized in walking from A to B ***R1***

[3 marks]

Examiners report

a. The fairly easy trigonometry challenged a large number of candidates.

b. Part (b) was very well done.

c. Satisfactory answers were very rarely seen for (c). Very few candidates realised that a minimum can occur at the beginning or end of an interval.

---

a. Using the definition of a derivative as $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, show that the derivative of $\frac{1}{2x+1}$ is $\frac{-2}{(2x+1)^2}$. [4]

b. Prove by induction that the n^{th} derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$. [9]

Markscheme

a. let $f(x) = \frac{1}{2x+1}$ and using the result $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$f'(x) = \lim_{h\to 0}\left(\frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}\right)$ ***M1A1***

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \frac{-2}{(2x+1)^2}$ ***AG***

[4 marks]

b. let $y = \frac{1}{2x+1}$

we want to prove that $\frac{d^n y}{dx^n} = (-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$

let $n = 1 \Rightarrow \frac{dy}{dx} = (-1)^1 \frac{2^1 1!}{(2x+1)^{1+1}}$ ***M1***

$\Rightarrow \frac{dy}{dx} = \frac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n = 1$ ***R1***

assume the result is true for $n = k$: $\frac{d^k y}{dx^k} = (-1)^k \frac{2^k k!}{(2x+1)^{k+1}}$ ***M1***

$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[(-1)^k \frac{2^k k!}{(2x+1)^{k+1}}\right]$ ***M1***

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$ ***(A1)***

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = (-1)^k 2^k k!(-k-1)(2x+1)^{-k-2} \times 2$ ***A1***

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = (-1)^{k+1} 2^{k+1}(k+1)!(2x+1)^{-k-2}$ ***(A1)***

$\Rightarrow \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = (-1)^{k+1}\frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$ ***A1***
hence if the result is true for $n = k$, it is true for $n = k + 1$
since the result is true for $n = 1$, the result is proved by mathematical induction ***R1***
Note: Only award final ***R1*** if all the ***M*** marks have been gained.

[9 marks]

Examiners report

a. Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k + 1$'.

b. Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k + 1$'.

Calculate the exact value of $\int_1^{\mathrm{e}} x^2 \ln x \mathrm{d}x$.

Markscheme

Recognition of integration by parts ***M1***

$\int x^2 \ln x \mathrm{d}x = \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^3}{3} \times \frac{1}{x}\mathrm{d}x$ ***A1A1***

$\quad = \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^2}{3}\mathrm{d}x$

$\quad = \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]$ ***A1***

$\Rightarrow \int_1^{\mathrm{e}} x^2 \ln x \mathrm{d}x = \left(\frac{\mathrm{e}^3}{3} - \frac{\mathrm{e}^3}{9}\right) - \left(0 - \frac{1}{9}\right) \quad \left(= \frac{2\mathrm{e}^3+1}{9}\right)$ ***A1***

[5 marks]

Examiners report

Most candidates recognised that a method of integration by parts was appropriate for this question. However, although a good number of correct answers were seen, a number of candidates made algebraic errors in the process. A number of students were also unable to correctly substitute the limits.

The function f is defined by

$$f(x) = \begin{cases} 1-2x, & x \le 2 \\ \frac{3}{4}(x-2)^2 - 3, & x > 2 \end{cases}$$

a. Determine whether or not f is continuous. [2]

b. The graph of the function g is obtained by applying the following transformations to the graph of f: [4]

a reflection in the y–axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Find $g(x)$.

Markscheme

a. $1-2(2) = -3$ and $\frac{3}{4}(2-2)^2 - 3 = -3$ ***A1***

both answers are the same, hence f is continuous (at $x = 2$) ***R1***

Note: ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

[2 marks]

b. reflection in the y-axis

$$f(-x) = \begin{cases} 1+2x, & x \ge -2 \\ \frac{3}{4}(x+2)^2 - 3, & x < -2 \end{cases} \quad \textit{(M1)}$$

Note: Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x) = \begin{cases} 2x-3, & x \ge 0 \\ \frac{3}{4}x^2 - 3, & x < 0 \end{cases} \quad \textit{(M1)A1A1}$$

Note: Award ***(M1)*** for attempting to substitute $(x-2)$ for X, or translating a graph along positive x-axis.
Award ***A1*** for the correct domains (this mark can be awarded independent of the ***M1***).
Award ***A1*** for the correct expressions.

[4 marks]

Examiners report

a. [N/A]

b. [N/A]

The function f is defined, for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, by $f(x) = 2\cos x + x\sin x$.

a. Determine whether f is even, odd or neither even nor odd. [3]

b. Show that $f''(0) = 0$. [2]

c. John states that, because $f''(0) = 0$, the graph of *f* has a point of inflexion at the point (0, 2) . Explain briefly whether John's statement is correct or not. [2]

Markscheme

a. $f(-x) = 2\cos(-x) + (-x)\sin(-x)$ ***M1***

$= 2\cos x + x\sin x \quad (= f(x))$ ***A1***

therefore *f* is even ***A1***

[3 marks]

b. $f'(x) = -2\sin x + \sin x + x\cos x \quad (= -\sin x + x\cos x)$ ***A1***

$f''(x) = -\cos x + \cos x - x\sin x \quad (= -x\sin x)$ ***A1***

so $f''(0) = 0$ ***AG***

[2 marks]

c. John's statement is incorrect because

either; there is a stationary point at (0, 2) and since *f* is an even function and therefore symmetrical about the *y*-axis it must be a maximum or a minimum

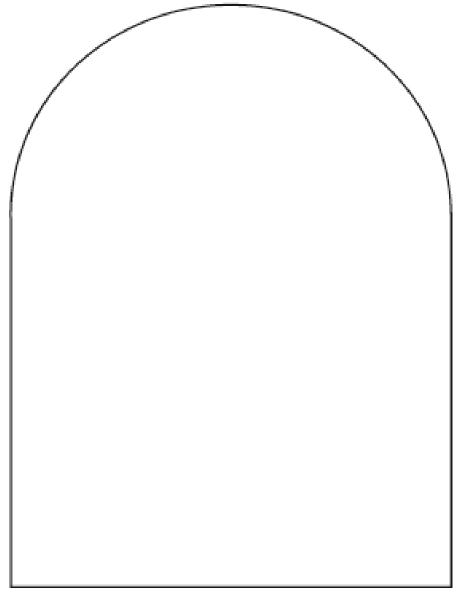
or; $f''(x)$ is even and therefore has the same sign either side of (0, 2) ***R2***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

A window is made in the shape of a rectangle with a semicircle of radius r metres on top, as shown in the diagram. The perimeter of the window is a constant P metres.



a.i. Find the area of the window in terms of P and r. [4]

a.ii. Find the width of the window in terms of P when the area is a maximum, justifying that this is a maximum. [5]

b. Show that in this case the height of the rectangle is equal to the radius of the semicircle. [2]

Markscheme

a.i. the width of the rectangle is $2r$ and let the height of the rectangle be h

$P = 2r + 2h + \pi r$ ***(A1)***

$A = 2rh + \frac{\pi r^2}{2}$ ***(A1)***

$h = \frac{\mathrm{P} - 2r - \pi r}{2}$

$A = 2r\left(\frac{\mathrm{P}-2r-\pi r}{2}\right) + \frac{\pi r^2}{2}$ $\left(= \mathrm{P}\,r - 2r^2 - \frac{\pi r^2}{2}\right)$ ***M1A1***

[4 marks]

a.ii. $\frac{\mathrm{d}A}{\mathrm{d}r} = \mathrm{P} - 4r - \pi r$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ ***M1***

$\Rightarrow r = \frac{\mathrm{P}}{4+\pi}$ ***(A1)***

hence the width is $\frac{2\mathrm{P}}{4+\pi}$ ***A1***

$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = -4 - \pi < 0$ ***R1***

hence maximum ***AG***

[5 marks]

b. **EITHER**

$h = \frac{\mathrm{P}-2r-\pi r}{2}$

$h = \frac{\mathrm{P} - \frac{2\mathrm{P}}{4+\pi} - \frac{\mathrm{P}\pi}{4+\pi}}{2}$ ***M1***

$h = \frac{4\mathrm{P}+\pi\mathrm{P}-2\mathrm{P}-\pi\mathrm{P}}{2(4+\pi)}$ ***A1***

$h = \frac{\mathrm{P}}{(4+\pi)} = r$ ***AG***

OR

$h = \frac{\mathrm{P} - 2r - \pi r}{2}$

$P = r(4 + \pi)$ ***M1***

$h = \frac{r(4+\pi) - 2r - \pi r}{2}$ ***A1***

$h = \frac{4r + \pi r - 2r - \pi r}{2} = r$ ***AG***

[2 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

A tranquilizer is injected into a muscle from which it enters the bloodstream.

The concentration C in mgl^{-1}, of tranquilizer in the bloodstream can be modelled by the function $C(t) = \frac{2t}{3+t^2},\ t \ge 0$ where t is the number of minutes after the injection.

Find the maximum concentration of tranquilizer in the bloodstream.

Markscheme

use of the quotient rule or the product rule ***M1***

$C'(t) = \frac{(3+t^2) \times 2 - 2t \times 2t}{\left(3+t^2\right)^2} \quad \left(= \frac{6-2t^2}{\left(3+t^2\right)^2}\right) \quad \text{or} \quad \frac{2}{3+t^2} - \frac{4t^2}{\left(3+t^2\right)^2}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator in the quotient rule, and ***A1*** for each correct term in the product rule.

attempting to solve $C'(t) = 0 \quad \text{for } t$ ***(M1)***

$t = \pm\sqrt{3} \quad (\text{minutes})$ ***A1***

$C\left(\sqrt{3}\right) = \frac{\sqrt{3}}{3} \quad \left(\mathrm{mgl}^{-1}\right) \quad \text{or equivalent.}$ ***A1***

[6 marks]

Examiners report

This question was generally well done. A significant number of candidates did not calculate the maximum value of C.

---

a. Show that $\cot \alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

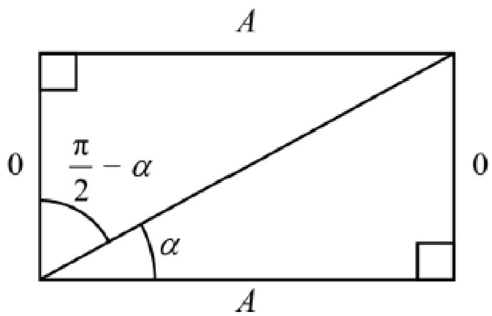
b. Hence find $\int_{\tan \alpha}^{\cot \alpha} \frac{1}{1+x^2} \mathrm{d}x,\ 0 < \alpha < \frac{\pi}{2}$. [4]

Markscheme

a. **EITHER**

use of a diagram and trig ratios

eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{A}{O}$ ***R1***

OR

use of $\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos\alpha}{\sin\alpha}$ ***R1***

THEN

$\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$ ***AG***

[1 mark]

b. $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\mathrm{d}x = [\arctan x]_{\tan\alpha}^{\cot\alpha}$ ***(A1)***

Note: Limits (or absence of such) may be ignored at this stage.

$= \arctan(\cot\alpha) - \arctan(\tan\alpha)$ ***(M1)***

$= \frac{\pi}{2} - \alpha - \alpha$ ***(A1)***

$= \frac{\pi}{2} - 2\alpha$ ***A1***

[4 marks]

Examiners report

a. This was generally well done.

b. This was generally well done. Some weaker candidates tried to solve part (b) through use of a substitution, though the standard result $\arctan x$ was well known. A small number used $\arctan x + c$ and went on to obtain an incorrect final answer.

Given that $\int_{-2}^{2} f(x)\,\mathrm{d}x = 10$ and $\int_{0}^{2} f(x)\,\mathrm{d}x = 12$, find

a. $\int_{-2}^{0} (f(x) + 2)\,\mathrm{d}x$. [4]

b. $\int_{-2}^{0} f(x+2)\,\mathrm{d}x$. [2]

Markscheme

a. $\int_{-2}^{0} f(x)\,\mathrm{d}x = 10 - 12 = -2$ ***(M1)(A1)***

$\int_{-2}^{0} 2\mathrm{d}x = [2x]_{-2}^{0} = 4$ ***A1***

$\int_{-2}^{0} (f(x)\ +2)\,\mathrm{d}x = 2$ ***A1***

[4 marks]

b. $\int_{-2}^{0} f(x+2)\,\mathrm{d}x = \int_{0}^{2} f(x)\,\mathrm{d}x$ ***(M1)***

= 12 ***A1***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

By using the substitution $t = \tan x$, find $\int \frac{\mathrm{d}x}{1+\sin^2 x}$.

Express your answer in the form $m\arctan(n\tan x) + c$, where m, n are constants to be determined.

Markscheme

EITHER

$x = \arctan t$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+t^2}$ ***A1***

OR

$t = \tan x$

$\frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 x$ ***(M1)***

$= 1 + \tan^2 x$ ***A1***

$= 1 + t^2$

THEN

$\sin x = \frac{t}{\sqrt{1+t^2}}$ ***(A1)***

Note: This ***A1*** is independent of the first two marks

$\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\frac{\mathrm{d}t}{1+t^2}}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to obtain integral in terms of t and $\mathrm{d}t$

$= \int \frac{\mathrm{d}t}{(1+t^2)+t^2} = \int \frac{\mathrm{d}t}{1+2t^2}$ ***A1***

$= \frac{1}{2}\int \frac{\mathrm{d}t}{\frac{1}{2}+t^2} = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}}\arctan\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$ ***A1***

$= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x\right)(+c)$ ***A1***

[8 marks]

Examiners report

[N/A]

Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point (1, 2) .

Markscheme

$5y^2 + 10xy\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0$ ***A1A1A1***

Note: Award ***A1A1*** for correct differentiation of $5xy^2$.
A1 for correct differentiation of $-2x^2$ and 18.

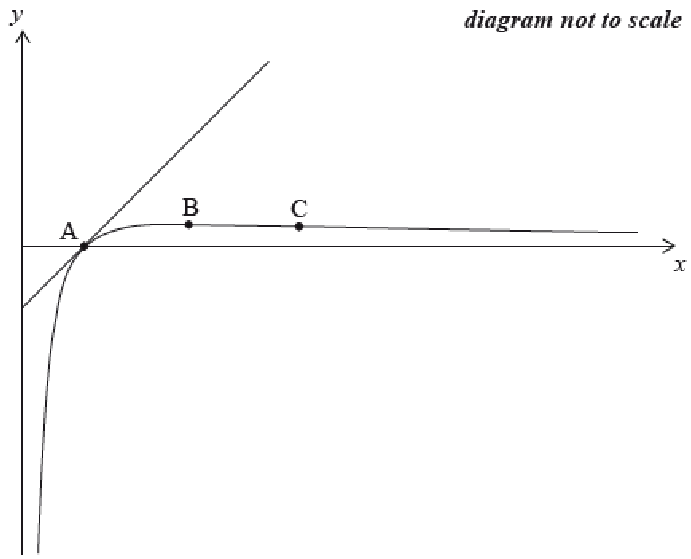
At the point (1, 2), $20 + 20\frac{\mathrm{d}y}{\mathrm{d}x} - 4 = 0$
$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{5}$ ***(A1)***
Gradient of normal $= \frac{5}{4}$ ***A1***
Equation of normal $y - 2 = \frac{5}{4}(x-1)$ ***M1***
$y = \frac{5}{4}x - \frac{5}{4} + \frac{8}{4}$
$y = \frac{5}{4}x + \frac{3}{4} \quad (4y = 5x + 3)$ ***A1***
[7 marks]

Examiners report

It was pleasing to see that a significant number of candidates understood that implicit differentiation was required and that they were able to make a reasonable attempt at this. A small number of candidates tried to make the equation explicit. This method will work, but most candidates who attempted this made either arithmetic or algebraic errors, which stopped them from gaining the correct answer.

Consider the function $f(x) = \frac{\ln x}{x},\ x > 0$.

The sketch below shows the graph of $y = f(x)$ and its tangent at a point A.



a. Show that $f'(x) = \frac{1-\ln x}{x^2}$. [2]

b. Find the coordinates of B, at which the curve reaches its maximum value. [3]

c. Find the coordinates of C, the point of inflexion on the curve. [5]

d. The graph of $y = f(x)$ crosses the x-axis at the point A. [4]

Find the equation of the tangent to the graph of f at the point A.

e. The graph of $y = f(x)$ crosses the x-axis at the point A. [7]

Find the area enclosed by the curve $y = f(x)$, the tangent at A, and the line $x = \mathrm{e}$.

Markscheme

a. $f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \frac{1-\ln x}{x^2}$ ***AG***

[2 marks]

b. $\frac{1-\ln x}{x^2} = 0$ has solution $x = \mathrm{e}$ ***M1A1***

$y = \frac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$

[3 marks]

c. $f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - 2x(1-\ln x)}{x^4}$ ***M1A1***

$= \frac{2\ln x - 3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where $f''(x) = 0$ ***M1***

so $x = \mathrm{e}^{\frac{3}{2}},\ y = \frac{3}{2}\mathrm{e}^{\frac{-3}{2}}$ ***A1A1***

C has coordinates $\left(\mathrm{e}^{\frac{3}{2}},\ \frac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

[5 marks]

d. $f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

$y = x + c$ ***(M1)***

through $(1, 0)$

equation is $y = x - 1$ ***A1***

[4 marks]

e. **METHOD 1**

area $= \int_1^{\mathrm{e}} x - 1 - \frac{\ln x}{x} \mathrm{d}x$ ***M1A1A1***

Note: Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$\int \frac{\ln x}{x} \mathrm{d}x = \frac{(\ln x)^2}{2} (+c)$ ***(M1)A1***

$\int (x - 1) \mathrm{d}x = \frac{x^2}{2} - x (+c)$ ***A1***

$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^{\mathrm{e}}$

$= \left(\frac{1}{2}\mathrm{e}^2 - \mathrm{e} - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}\mathrm{e}^2 - \mathrm{e}$ ***A1***

METHOD 2

area = area of triangle $- \int_1^e \frac{\ln x}{x} \mathrm{d}x$ ***M1A1***

Note: ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$\int \frac{\ln x}{x} \mathrm{d}x = \frac{(\ln x)^2}{2} (+c)$ ***(M1)A1***

area of triangle $= \frac{1}{2}(e - 1)(e - 1)$ ***M1A1***

$\frac{1}{2}(e - 1)(e - 1) - \left(\frac{1}{2}\right) = \frac{1}{2}\mathrm{e}^2 - \mathrm{e}$ ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

The region enclosed between the curves $y = \sqrt{x}\mathrm{e}^x$ and $y = \mathrm{e}\sqrt{x}$ is rotated through 2π about the x-axis. Find the volume of the solid obtained.

Markscheme

$\sqrt{x}\mathrm{e}^x = \mathrm{e}\sqrt{x} \Rightarrow x = 0$ or 1 ***(A1)***

attempt to find $\int y^2 \mathrm{d}x$ ***M1***

$V_1 = \pi \int_0^1 \mathrm{e}^2 x \mathrm{d}x$

$= \pi \left[\frac{1}{2}\mathrm{e}^2 x^2\right]_0^1$

$= \frac{\pi \mathrm{e}^2}{2}$ ***A1***

$V_2 = \pi \int_0^1 x\mathrm{e}^{2x} \mathrm{d}x$

$= \pi \left(\left[\frac{1}{2}x\mathrm{e}^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}\mathrm{e}^{2x} \mathrm{d}x\right)$ ***M1A1***

Note: Award ***M1*** for attempt to integrate by parts.

$= \frac{\pi e^2}{2} - \pi\left[\frac{1}{4}e^{2x}\right]_0^1$

finding difference of volumes ***M1***

volume$= V_1 - V_2$

$= \pi\left[\frac{1}{4}e^{2x}\right]_0^1$

$= \frac{1}{4}\pi(e^2 - 1)$ ***A1***

[7 marks]

Examiners report

While only a minority of candidates achieved full marks in this question, many candidates made good attempts. Quite a few candidates obtained the limits correctly and many realized a square was needed in the integral, though a number of them subtracted then squared rather than squaring and then subtracting. There was evidence that quite a few knew about integration by parts. One common mistake was to have 2π, rather than π in the integral.

The curve C is given by $y = \frac{x\cos x}{x+\cos x}$, for $x \geqslant 0$.

a. Show that $\frac{dy}{dx} = \frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2},\ x \geqslant 0$. [4]

b. Find the equation of the tangent to C at the point $\left(\frac{\pi}{2}, 0\right)$. [3]

Markscheme

a. $\frac{dy}{dx} = \frac{(x+\cos x)(\cos x - x\sin x) - x\cos x(1-\sin x)}{(x+\cos x)^2}$ ***M1A1A1***

Note: Award ***M1*** for attempt at differentiation of a quotient and a product condoning sign errors in the quotient formula and the trig differentiations, ***A1*** for correct derivative of "*u*", ***A1*** for correct derivative of "*v*".

$= \frac{x\cos x + \cos^2 x - x^2\sin x - x\cos x\sin x - x\cos x + x\cos x\sin x}{(x+\cos x)^2}$ ***A1***

$= \frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2}$ ***AG***

[4 marks]

b. the derivative has value –1 ***(A1)***

the equation of the tangent line is $(y-0) = (-1)\left(x - \frac{\pi}{2}\right)\ \left(y = \frac{\pi}{2} - x\right)$ ***M1A1***

[3 marks]

Examiners report

a. The majority of candidates earned significant marks on this question. The product rule and the quotient rule were usually correctly applied, but a few candidates made an error in differentiating the denominator, obtaining $-\sin x$ rather than $1-\sin x$. A disappointing number of candidates failed to calculate the correct gradient at the specified point.

b. The majority of candidates earned significant marks on this question. The product rule and the quotient rule were usually correctly applied, but a few candidates made an error in differentiating the denominator, obtaining $-\sin x$ rather than $1-\sin x$. A disappointing number of candidates failed to calculate the correct gradient at the specified point.

---

a. Find $\int (1+\tan^2 x)\mathrm{d}x$. [2]

b. Find $\int \sin^2 x\mathrm{d}x$. [3]

Markscheme

a. $\int (1+\tan^2 x)\mathrm{d}x = \int \sec^2 x\mathrm{d}x = \tan x(+c)$ ***M1A1***

[2 marks]

b. $\int \sin^2 x\mathrm{d}x = \int \frac{1-\cos 2x}{2}\mathrm{d}x$ ***M1A1***

$= \frac{x}{2} - \frac{\sin 2x}{4}(+c)$ ***A1***

Note: Allow integration by parts followed by trig identity.

Award ***M1*** for parts, ***A1*** for trig identity, ***A1*** final answer.

[3 marks]

Total [5 marks]

Examiners report

a. Some correct answers but too many candidates had a poor approach and did not use the trig identity.

b. Same as (a).

By using the substitution $u = 1+\sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}}\mathrm{d}x$.

Markscheme

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ ***A1***

$\mathrm{d}x = 2(u-1)\mathrm{d}u$

Note: Award the ***A1*** for any correct relationship between $\mathrm{d}x$ and $\mathrm{d}u$.

$\int \frac{\sqrt{x}}{1+\sqrt{x}}\mathrm{d}x = 2\int \frac{(u-1)^2}{u}\mathrm{d}u$ ***(M1)A1***

Note: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u .

$= 2\int u - 2 + \frac{1}{u}\mathrm{d}u$ ***(A1)***

$= u^2 - 4u + 2\ln u(+C)$ ***A1***

$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$ ***A1***

Note: Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

[6 marks]

Examiners report

Many candidates worked through this question successfully. A significant minority either made algebraic mistakes with the substitution or tried to work with an integral involving both x and u.

Let $y = \sin^2\theta,\ 0 \leqslant \theta \leqslant \pi$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ [2]

b. Hence find the values of θ for which $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2y$. [5]

Markscheme

a. attempt at chain rule or product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\sin\theta\cos\theta$ ***A1***

[2 marks]

b. $2\sin\theta\cos\theta = 2\sin^2\theta$

$\sin\theta = 0$ ***(A1)***

$\theta = 0, \pi$ ***A1***

obtaining $\cos\theta = \sin\theta$ ***(M1)***

$\tan\theta = 1$ ***(M1)***

$\theta = \frac{\pi}{4}$ ***A1***

[5 marks]

Examiners report

a. [N/A]

b. [N/A]

A particle moves along a straight line. Its displacement, s metres, at time t seconds is given by $s = t + \cos 2t,\ t \geqslant 0$. The first two times when the particle is at rest are denoted by t_1 and t_2, where $t_1 < t_2$.

a. Find t_1 and t_2. [5]

b. Find the displacement of the particle when $t = t_1$ [2]

Markscheme

a. $s = t + \cos 2t$

$\frac{\mathrm{d}s}{\mathrm{d}t} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s),\ t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

b. $s = \frac{\pi}{12} + \cos\frac{\pi}{6}\ \left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

---

a. Using the substitution $x = \tan\theta$ show that $\int\limits_0^1 \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x = \int\limits_0^{\frac{\pi}{4}} \cos^2\theta \mathrm{d}\theta$. [4]

b. Hence find the value of $\int\limits_0^1 \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x$. [3]

Markscheme

a. let $x = \tan\theta$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ ***(A1)***

$\int \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x = \int \frac{\sec^2\theta}{\left(\tan^2\theta+1\right)^2}\mathrm{d}\theta$ ***M1***

Note: The method mark is for an attempt to substitute for both x and $\mathrm{d}x$.

$= \int \frac{1}{\sec^2\theta}\mathrm{d}\theta$ (or equivalent) ***A1*** when $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{4}$ ***M1***

$\int\limits_0^{\frac{\pi}{4}} \cos^2\theta \mathrm{d}\theta$ ***AG***

[4 marks]

b. $\left(\int\limits_0^1 \frac{1}{(x^2+1)^2}\mathrm{d}x = \int\limits_0^{\frac{\pi}{4}} \cos^2\theta \mathrm{d}\theta\right) = \frac{1}{2}\int\limits_0^{\frac{\pi}{4}} (1+\cos 2\theta)\,\mathrm{d}\theta$ ***M1***

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$ ***A1***

$= \frac{\pi}{8} + \frac{1}{4}$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

Use the substitution $u = \ln x$ to find the value of $\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{\mathrm{d}x}{x\ln x}$.

Markscheme

METHOD 1

$\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{\mathrm{d}x}{x\ln x} = [\ln(\ln x)]_{\mathrm{e}}^{\mathrm{e}^2}$ ***(M1)A1***

$= \ln(\ln \mathrm{e}^2) - \ln(\ln \mathrm{e}) \quad (= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

METHOD 2

$u = \ln x,\ \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

$= \int_1^2 \frac{\mathrm{d}u}{u}$ ***A1***

Note: Condone absent or incorrect limits here.

$= [\ln u]_1^2$ or equivalent in $x (= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

Examiners report

[N/A]

---

a. Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}},\ 0 \leqslant x \leqslant \pi$. [2]

b. Find a similar expression for $\sin\frac{1}{2}x,\ 0 \leqslant x \leqslant \pi$. [2]

c. Hence find the value of $\int_0^{\frac{\pi}{2}}\left(\sqrt{1+\cos x}+\sqrt{1-\cos x}\right)\mathrm{d}x$. [4]

Markscheme

a. $\cos x = 2\cos^2\frac{1}{2}x - 1$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \leqslant x \leqslant \pi$ ***R1***

$\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

b. $\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

c. $\sqrt{2}\int_0^{\frac{\pi}{2}}\cos\frac{1}{2}x + \sin\frac{1}{2}x\mathrm{d}x$ ***A1***

$= \sqrt{2}\left[2\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***A1***

$= 2\sqrt{2}$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Paint is poured into a tray where it forms a circular pool with a uniform thickness of 0.5 cm. If the paint is poured at a constant rate of 4 cm^3s^{-1}, find the rate of increase of the radius of the circle when the radius is 20 cm.

Markscheme

$V = 0.5\pi r^2$ ***(A1)***

EITHER

$\frac{dV}{dr} = \pi r$ ***A1***

$\frac{dV}{dt} = 4$ ***(A1)***

applying chain rule ***M1***

for example $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

OR

$\frac{dV}{dt} = \pi r\frac{dr}{dt}$ ***M1A1***

$\frac{dV}{dt} = 4$ ***(A1)***

THEN

$\frac{dr}{dt} = 4 \times \frac{1}{\pi r}$ ***A1*** when $r = 20$, $\frac{dr}{dt} = \frac{4}{20\pi}$ or $\frac{1}{5\pi}$ $(\mathrm{cm\,s^{-1}})$ ***A1***

Note: Allow h instead of 0.5 up until the final ***A1***.

[6 marks]

Examiners report

There was a large variety of methods used in this question, with most candidates choosing to implicitly differentiate the expression for volume in terms of r.

Consider two functions f and g and their derivatives f' and g'. The following table shows the values for the two functions and their derivatives at $x = 1$, 2 and 3.

x	1	2	3
$f(x)$	3	1	1
$f'(x)$	1	4	2
$g(x)$	2	1	4
$g'(x)$	4	2	3

Given that $p(x) = f(x)g(x)$ and $h(x) = g \circ f(x)$, find

a. $p'(3)$; [2]

b. $h'(2)$. [4]

Markscheme

a. $p'(3) = f'(3)g(3) + g'(3)f(3)$ ***(M1)***

Note: Award ***M1*** if the derivative is in terms of x or 3.

$= 2 \times 4 + 3 \times 1$

$= 11$ ***A1***

[2 marks]

b. $h'(x) = g'(f(x))\, f'(x)$ ***(M1)(A1)***

$h'(2) = g'(1)f'(2)$ ***A1***

$= 4 \times 4$

$= 16$ ***A1***

[4 marks]

Total [6 marks]

Examiners report

a. This was a problem question for many candidates. Some quite strong candidates, on the evidence of their performance on other questions, did not realise that ‘composite functions’ and ‘functions of a function’ were the same thing, and therefore that the chain rule applied.

b. This was a problem question for many candidates. Some quite strong candidates, on the evidence of their performance on other questions, did not realise that ‘composite functions’ and ‘functions of a function’ were the same thing, and therefore that the chain rule applied.

The region bounded by the curve $y = \frac{\ln(x)}{x}$ and the lines $x = 1$, $x = e$, $y = 0$ is rotated through 2π radians about the x-axis.

Find the volume of the solid generated.

Markscheme

METHOD 1

$V = \pi \int_1^{\mathrm{e}} \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$ ***M1***

Integrating by parts:

$u = (\ln x)^2,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\ln x}{x},\ v = -\frac{1}{x}$

$\Rightarrow V = \pi\left(-\frac{(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2}\mathrm{d}x\right)$ ***A1***

$u = \ln x,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x},\ v = -\frac{1}{x}$

$\therefore \int \frac{\ln x}{x^2}\mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2}\mathrm{d}x = -\frac{\ln x}{x} - \frac{1}{x}$ ***A1***

$\therefore V = \pi\left[-\frac{(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\right]_1^{\mathrm{e}}$

$= 2\pi - \frac{5\pi}{e}$ ***A1***

[6 marks]

METHOD 2

$V = \pi \int_1^{\mathrm{e}} \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$ ***M1***

Let $\ln x = u \Rightarrow x = \mathrm{e}^u,\ \frac{\mathrm{d}x}{x} = \mathrm{d}u$ ***(M1)***

$\int \left(\frac{\ln x}{x}\right)^2 \mathrm{d}x = \int \frac{u^2}{\mathrm{e}^u}\mathrm{d}u = \int \mathrm{e}^{-u}u^2\mathrm{d}u = -\mathrm{e}^{-u}u^2 + 2\int \mathrm{e}^{-u}u\mathrm{d}u$ ***A1***

$= -\mathrm{e}^{-u}u^2 + 2\left(-\mathrm{e}^{-u}u + \int \mathrm{e}^{-u}\mathrm{d}u\right) = -\mathrm{e}^{-u}u^2 - 2\mathrm{e}^{-u}u - 2\mathrm{e}^{-u}$

$= -\mathrm{e}^{-u}(u^2 + 2u + 2)$ ***A1***

When $x = \mathrm{e}$, $u = 1$. When $x = 1$, $u = 0$.

$\therefore$ Volume $= \pi\left[-\mathrm{e}^{-u}(u^2+2u+2)\right]_0^1$ ***M1***

$= \pi(-5\mathrm{e}^{-1}+2)\ \left(= 2\pi - \frac{5\pi}{\mathrm{e}}\right)$ ***A1***

[6 marks]

Examiners report

Only the best candidates were able to make significant progress with this question. It was disappointing to see that many candidates could not state that the formula for the required volume was $\pi\int_1^e\left(\frac{\ln x}{x}\right)^2\mathrm{d}x$. Of those who could, very few either attempted integration by parts or used an appropriate substitution.

Consider the curve $y = \frac{1}{1-x},\ x \in \mathbb{R},\ x \neq 1.$

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Determine the equation of the normal to the curve at the point $x = 3$ in the form $ax + by + c = 0$ where $a,\ b,\ c \in \mathbb{Z}$. [4]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = (1-x)^{-2}\quad\left(=\frac{1}{(1-x)^2}\right)$ ***(M1)A1***

[2 marks]

b. gradient of Tangent $= \frac{1}{4}$ ***(A1)***

gradient of Normal $= -4$ ***(M1)***

$y + \frac{1}{2} = -4(x-3)$ or attempt to find c in $y = mx + c$ ***M1***

$8x + 2y - 23 = 0$ ***A1***

[4 marks]

Total [6 marks]

Examiners report

a. [N/A]
b. [N/A]

The function f is given by $f(x) = x\mathrm{e}^{-x}\ (x \geqslant 0)$.

a(i)(ii) Find an expression for $f'(x)$. [3]

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$.

b. Find an expression for $f''(x)$ and hence show the point A is a maximum. [3]

c. Find the coordinates of B, the point of inflexion. [2]

d. The graph of the function g is obtained from the graph of f by stretching it in the x-direction by a scale factor 2. [5]

(i) Write down an expression for $g(x)$.

(ii) State the coordinates of the maximum C of g.

(iii) Determine the x-coordinates of D and E, the two points where $f(x) = g(x)$.

e. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. [4]

f. Find an exact value for the area of the region bounded by the curve $y = g(x)$, the x-axis and the line $x = 1$. [3]

Markscheme

a(i)(i) $f'(x) = \mathrm{e}^{-x} - x\mathrm{e}^{-x}$ ***M1A1***

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1,\ \mathrm{e}^{-1}\right)$ ***A1***

[3 marks]

b. $f''(x) = -\mathrm{e}^{-x} - \mathrm{e}^{-x} + x\mathrm{e}^{-x}\ \ (= -\mathrm{e}^{-x}(2 - x))$ ***A1***

substituting $x = 1$ into $f''(x)$ ***M1***

$f''(1)\ \left(= -\mathrm{e}^{-1}\right) < 0$ hence maximum ***R1AG***

[3 marks]

c. $f''(x) = 0\ (\Rightarrow x = 2)$ ***M1***

coordinates $\left(2,\ 2\mathrm{e}^{-2}\right)$ ***A1***

[2 marks]

d. (i) $g(x) = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$ ***A1***

(ii) coordinates of maximum $\left(2,\ \mathrm{e}^{-1}\right)$ ***A1***

(iii) equating $f(x) = g(x)$ and attempting to solve $x\mathrm{e}^{-x} = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$

$\Rightarrow x\left(2\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{x}\right) = 0$ ***(A1)***

$\Rightarrow x = 0$ ***A1***

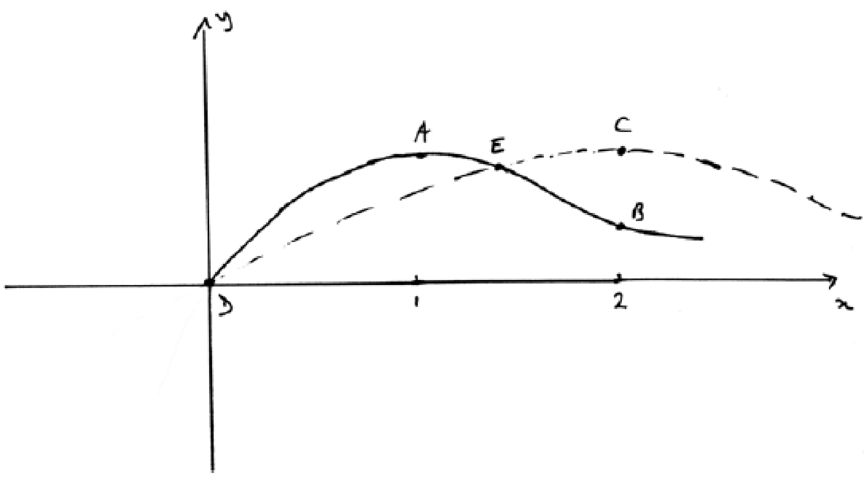
or $2\mathrm{e}^{\frac{x}{2}} = \mathrm{e}^{x}$

$\Rightarrow \mathrm{e}^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2\ \ (\ln 4)$ ***A1***

Note: Award first ***(A1)*** only if factorisation seen or if two correct solutions are seen.

e.



A4

Note: Award ***A1*** for shape of f, including domain extending beyond $x = 2$.
Ignore any graph shown for $x < 0$.
Award ***A1*** for A and B correctly identified.
Award ***A1*** for shape of g, including domain extending beyond $x = 2$.
Ignore any graph shown for $x < 0$. Allow follow through from f.
Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

f. $A = \int_0^1 \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}\mathrm{d}x$ ***M1***

$= \left[-x\mathrm{e}^{-\frac{x}{2}}\right]_0^1 - \int_0^1 -\mathrm{e}^{-\frac{x}{2}}\mathrm{d}x$ ***A1***

Note: Condone absence of limits or incorrect limits.

$= -\mathrm{e}^{-\frac{1}{2}} - \left[2\mathrm{e}^{-\frac{x}{2}}\right]_0^1$

$= -\mathrm{e}^{-\frac{1}{2}} - \left(2\mathrm{e}^{-\frac{1}{2}} - 2\right) = 2 - 3\mathrm{e}^{-\frac{1}{2}}$ ***A1***

[3 marks]

Examiners report

a(i)(i)Part a) proved to be an easy start for the vast majority of candidates.

b. Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.
Part c) again proved to be an easily earned 2 marks.

c. Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.
Part c) again proved to be an easily earned 2 marks.

d. Many candidates lost their way in part d). A variety of possibilities for $g(x)$ were suggested, commonly $2x\mathrm{e}^{-2x}$, $\frac{x\mathrm{e}^{-1}}{2}$ or similar variations. Despite section ii) being worth only one mark, (and 'state' being present in the question), many laborious attempts at further differentiation were seen. Part diii was usually answered well by those who gave the correct function for $g(x)$.

e. Part e) was also answered well by those who had earned full marks up to that point.

f. While the integration by parts technique was clearly understood, it was somewhat surprising how many careless slips were seen in this part of the question. Only a minority gained full marks for part f).

If $f(x) = x - 3x^{\frac{2}{3}},\ x > 0$,

(a) find the x-coordinate of the point P where $f'(x) = 0$;

(b) determine whether P is a maximum or minimum point.

Markscheme

(a) $f'(x) = 1 - \frac{2}{x^{\frac{1}{3}}}$ ***A1***

$\Rightarrow 1-\frac{2}{x^{\frac{1}{3}}}=0 \Rightarrow x^{\frac{1}{3}}=2 \Rightarrow x=8$ ***A1***

(b) $f''(x)=\frac{2}{3x^{\frac{4}{3}}}$ ***A1***

$f''(8)>0 \Rightarrow$ at $x=8,\ f(x)$ has a minimum. ***M1A1***

[5 marks]

Examiners report

Most candidates were able to correctly differentiate the function and find the point where $f'(x)=0$. They were less successful in determining the nature of the point.

The normal to the curve $x\mathrm{e}^{-y}+\mathrm{e}^{y}=1+x$, at the point ($c$, ln c), has a y-intercept c^2+1.

Determine the value of c.

Markscheme

EITHER

differentiating implicitly:

$1\times\mathrm{e}^{-y}-x\mathrm{e}^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}=1$ ***M1A1***

at the point (c, ln c)

$\frac{1}{c}-c\times\frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}x}+c\frac{\mathrm{d}y}{\mathrm{d}x}=1$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{c}\quad(c\neq 1)$ ***(A1)***

OR

reasonable attempt to make expression explicit ***(M1)***

$x\mathrm{e}^{-y}+\mathrm{e}^{y}=1+x$

$x+\mathrm{e}^{2y}=\mathrm{e}^{y}(1+x)$

$\mathrm{e}^{2y}-\mathrm{e}^{y}(1+x)+x=0$

$(\mathrm{e}^{y}-1)(\mathrm{e}^{y}-x)=0$ ***(A1)***

$\mathrm{e}^{y}=1,\ \mathrm{e}^{y}=x$

$y=0,\ y=\ln x$ ***A1***

Note: Do not penalize if $y = 0$ not stated.

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}$

gradient of tangent $=\frac{1}{c}$ ***A1***

Note: If candidate starts with $y=\ln x$ with no justification, award ***(M0)(A0)A1A1***.

THEN

the equation of the normal is

$y - \ln c = -c(x - c)$ ***M1***

$x = 0,\ y = c^2 + 1$

$c^2 + 1 - \ln c = c^2$ ***(A1)***

$\ln c = 1$

$c = \mathrm{e}$ ***A1***

[7 marks]

Examiners report

This was the first question to cause the majority of candidates a problem and only the better candidates gained full marks. Weaker candidates made errors in the implicit differentiation and those who were able to do this often were unable to simplify the expression they gained for the gradient of the normal in terms of c; a significant number of candidates did not know how to simplify the logarithms appropriately.

The curve *C* is given implicitly by the equation $\frac{x^2}{y} - 2x = \ln y$ for $y > 0$.

a. Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x* and *y*. [4]

b. Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on *C* where *y* = 1 and $x > 0$. [2]

Markscheme

a. attempt at implicit differentiation ***M1***

EITHER

$\frac{2x}{y} - \frac{x^2}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - 2 = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for each side.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{2x}{y} - 2}{\frac{1}{y} + \frac{x^2}{y^2}} \left(= \frac{2xy - 2y^2}{x^2 + y}\right)$ ***A1***

OR

after multiplication by *y*

$2x - 2y - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\ln y + y\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for each side.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(x-y)}{1 + 2x + \ln y}$ ***A1***

[4 marks]

b. for $y = 1,\ x^2 - 2x = 0$

$x = (0 \text{ or})\ 2$ ***A1***

for $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{5}$ ***A1***

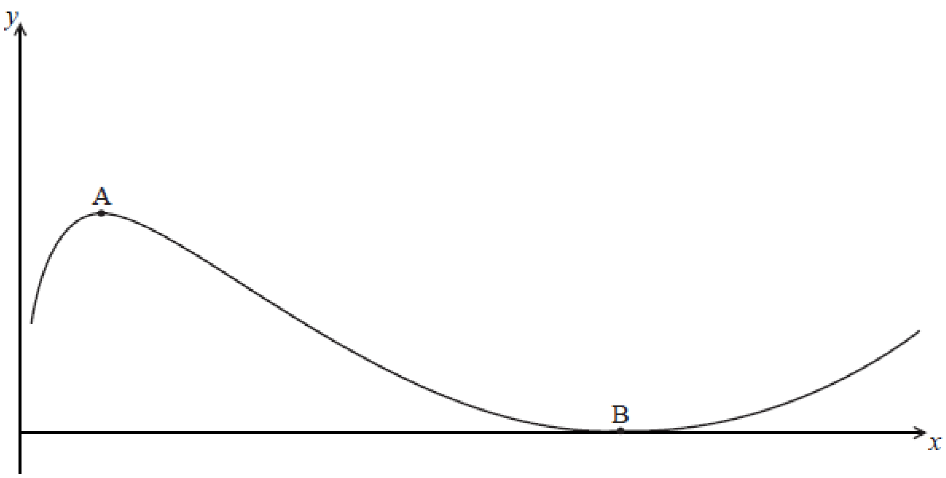
[2 marks]

Examiners report

a. Most candidates were familiar with the concept of implicit differentiation and the majority found the correct derivative function. In part (b), a significant number of candidates didn't realise that the value of *x* was required.

b. Most candidates were familiar with the concept of implicit differentiation and the majority found the correct derivative function. In part (b), a significant number of candidates didn't realise that the value of *x* was required.

The diagram shows the graph of the function defined by $y = x(\ln x)^2$ for $x > 0$.



The function has a local maximum at the point A and a local minimum at the point B.

a. Find the coordinates of the points A and B. [5]

b. Given that the graph of the function has exactly one point of inflexion, find its coordinates. [3]

Markscheme

a. $f'(x) = (\ln x)^2 + \frac{2x\ln x}{x} \left(= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)\right)$ ***M1A1***

$f'(x) = 0\ (\Rightarrow x = 1,\ x = e^{-2})$ ***M1***

Note: Award ***M1*** for an attempt to solve $f'(x) = 0$.

$A(e^{-2},\ 4e^{-2})$ **and** $B(1, 0)$ ***A1A1***

Note: The final ***A1*** is independent of prior working.

[5 marks]

b. $f''(x) = \frac{2}{x}(\ln x + 1)$ ***A1***

$f''(x) = 0\ \left(\Rightarrow x = e^{-1}\right)$ ***(M1)*** inflexion point $(e^{-1},\ e^{-1})$ ***A1***

Note: ***M1*** for attempt to solve $f''(x) = 0$.

[3 marks]

Examiners report

a. This was answered very well. Candidates are very familiar with this type of question. Some lost a couple of marks by failing to find their final y coordinates, though only the weakest struggled with differentiation and so made little progress.

b. This was answered very well. Candidates are very familiar with this type of question. Some lost a couple of marks by failing to find their final y coordinates, though only the weakest struggled with differentiation and so made little progress.

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$
$g(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

a. Sketch the graph of $y = h(x)$. [2]

b. Find an expression for the composite function $h \circ g(x)$ and state its domain. [2]

c. Given that $f(x) = h(x) + h \circ g(x)$, [7]

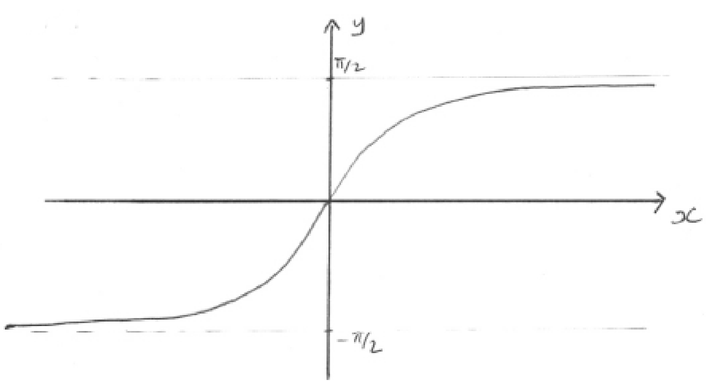
(i) find $f'(x)$ in simplified form;
(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

d. Nigel states that f is an odd function and Tom argues that f is an even function. [3]

(i) State who is correct and justify your answer.
(ii) Hence find the value of $f(x)$ for $x < 0$.

Markscheme

a.



A1A1

Note: ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

b. $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of $g: x \in \circ,\ x \neq 0$ ***A1***

[2 marks]

c. (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ ***M1A1***

$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$ ***(A1)***

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$

$= 0$ ***A1***

(ii) **METHOD 1**

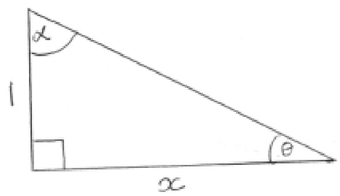
f is a constant ***R1***

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ ***M1A1***

$= \frac{\pi}{2}$ ***AG***

METHOD 2



from diagram

$\theta = \arctan\frac{1}{x}$ ***A1***

$\alpha = \arctan x$ ***A1***

$\theta + \alpha = \frac{\pi}{2}$ ***R1***

hence $f(x) = \frac{\pi}{2}$ ***AG***

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ ***M1***

$= \frac{x+\frac{1}{x}}{1-x\left(\frac{1}{x}\right)}$ ***A1***

denominator = 0, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) ***R1***

[7 marks]

d. (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore *f* is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The function f is defined by $f(x) = \mathrm{e}^x \sin x$.

a. Show that $f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$. [3]

b. Obtain a similar expression for $f^{(4)}(x)$. [4]

c. Suggest an expression for $f^{(2n)}(x)$, $n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction. [8]

Markscheme

a. $f'(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$ ***A1***

$f''(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x + \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$ ***A1***

$= 2\mathrm{e}^x \cos x$ ***A1***

$= 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ ***AG***

[3 marks]

b. $f'''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$f^{(4)}(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) - 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4\mathrm{e}^x \sin(x + \pi)$ ***A1***

[4 marks]

c. the conjecture is that

$f^{(2n)}(x) = 2^n \mathrm{e}^x \sin\left(x + \frac{n\pi}{2}\right)$ ***A1***

for $n = 1$, this formula gives

$f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ which is correct ***A1***

let the result be true for $n = k$, $\left(i.e.\ f^{(2k)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)\right)$ ***M1***

consider $f^{(2k+1)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***M1***

$f^{(2(k+1))}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) - 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \sin\left(x + \frac{(k+1)\pi}{2}\right)$ ***A1***

therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$

the result is proved by induction. ***R1***

Note: Award the final ***R1*** only if the two ***M*** marks have been awarded.

[8 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function f is defined by $f(x) = x\mathrm{e}^{2x}$.

It can be shown that $f^{(n)}(x) = (2^n x + n2^{n-1})\mathrm{e}^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(a) By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

(b) Show that f has a point of inflexion Q at $x = -1$.

(c) Determine the intervals on the domain of f where f is

(i) concave up;

(ii) concave down.

(d) Sketch f , clearly showing any intercepts, asymptotes and the points P and Q.

(e) Use mathematical induction to prove that $f^{(n)}(x) = (2^n x + n2^{n-1})\mathrm{e}^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}$ represents the n^{th} derivative of $f(x)$.

Markscheme

(a) $f'(x) = (1 + 2x)\mathrm{e}^{2x}$ ***A1***

$f'(x) = 0$ ***M1***

$\Rightarrow (1 + 2x)\mathrm{e}^{2x} = 0 \Rightarrow x = -\frac{1}{2}$ ***A1***

$f''(x) = (2^2 x + 2 \times 2^{2-1})\mathrm{e}^{2x} = (4x + 4)\mathrm{e}^{2x}$ ***A1***

$f''\left(-\frac{1}{2}\right) = \frac{2}{\mathrm{e}}$ ***A1***

$\frac{2}{\mathrm{e}} > 0 \Rightarrow$ at $x = -\frac{1}{2}$, $f(x)$ has a minimum. ***R1***

$\mathrm{P}\left(-\frac{1}{2}, -\frac{1}{2\mathrm{e}}\right)$ ***A1***

[7 marks]

(b) $f''(x) = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$ ***M1A1***

Using the 2nd derivative $f''\left(-\frac{1}{2}\right) = \frac{2}{\mathrm{e}}$ and $f''(-2) = -\frac{4}{\mathrm{e}^4}$, ***M1A1***

the sign change indicates a point of inflexion. ***R1***

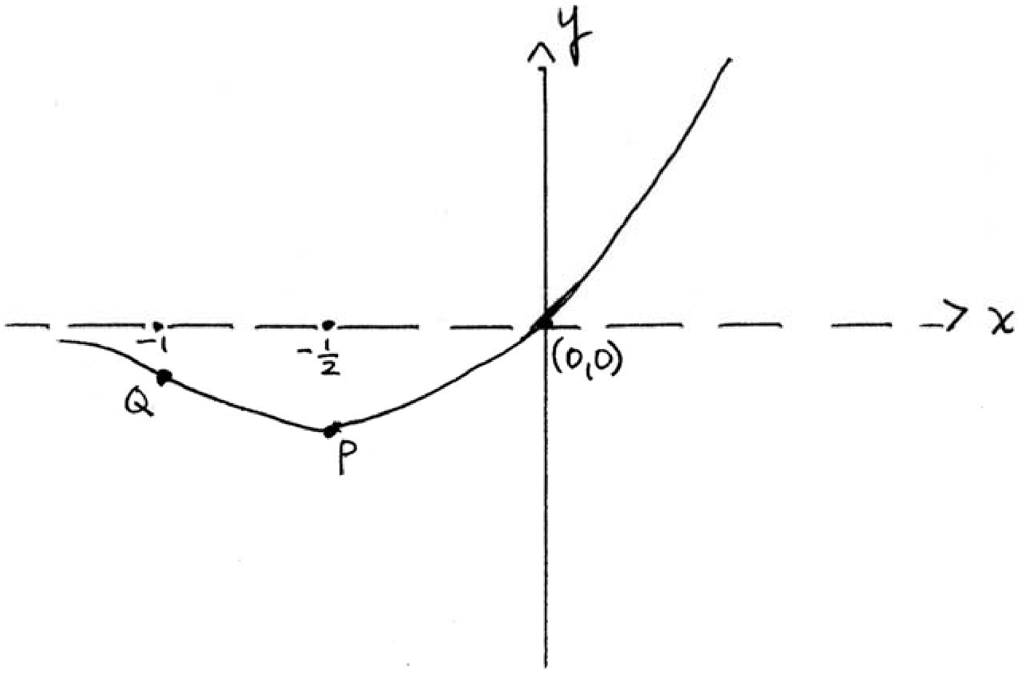
[5 marks]

(c) (i) $f(x)$ is concave up for $x > -1$. ***A1***

(ii) $f(x)$ is concave down for $x < -1$. ***A1***

[2 marks]

(d)



A1A1A1A1

Note: Award *A1* for P and Q, with Q above P,

A1 for asymptote at $y = 0$,

A1 for (0, 0) ,

A1 for shape.

[4 marks]

(e) Show true for $n = 1$ ***(M1)***

$f'(x) = e^{2x} + 2xe^{2x}$ ***A1***

$= e^{2x}(1 + 2x) = (2x + 2^0)e^{2x}$

Assume true for $n = k$, $i.e.$ $f^{(k)}x = (2^k x + k \times 2^{k-1})e^{2x}, k \geqslant 1$ ***M1A1***

Consider $n = k + 1$, $i.e.$ an attempt to find $\frac{d}{dx}\left(f^k(x)\right)$. ***M1***

$F^{(k+1)}(x) = 2^k e^{2x} + 2e^{2x}(2^k x + k \times 2^{k-1})$ ***A1***

$= \left(2^k + 2(2^k x + k \times 2^{k-1})\right) e^{2x}$

$= (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})e^{2x}$

$= (2^{k+1}x + 2^k + k \times 2^k)e^{2x}$ ***A1***

$= \left(2^{k+1}x + (k+1)\, 2^k\right) e^{2x}$ ***A1***

$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k + 1$, and since true for $n = 1$, result proved by mathematical induction $\forall n \in \mathbb{Z}^+$

Note: Only award ***R1*** if a reasonable attempt is made to prove the $(k+1)^{\text{th}}$ step.

[9 marks]

Total [27 marks]

Examiners report

This was the most accessible question in section B for these candidates. A majority of candidates produced partially correct answers to part (a), but a significant number struggled with demonstrating that the point is a minimum, despite the hint being given in the question. Part (b) started quite successfully but many students were unable to prove it is a point of inflexion or, more commonly, did not attempt to justify it. Correct answers were often seen for part (c). Part (d) was dependent on the successful completion of the first three parts. If candidates made errors in earlier parts, this often became obvious when they came to sketch the curve. However, few candidates realised that this part was a good way of checking that the above answers were at least consistent. The quality of curve sketching was rather weak overall, with candidates not marking points appropriately and not making features such as asymptotes clear. It is not possible to tell to what extent this was an effect of candidates not having a calculator, but it should be noted that asking students to sketch curves without a calculator will continue to appear on non-calculator papers. In part (e) the basic idea of proof by induction had clearly been taught with a significant majority of students understanding this. However, many candidates did not understand that they had to differentiate again to find the result for $(k+1)$.

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x),\ n \in \mathbb{Z}^+$.

a. Determine whether f_n is an odd or even function, justifying your answer. [2]

b. By using mathematical induction, prove that [8]

$$f_n(x) = \frac{\sin 2^{n+1}x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2} \text{ where } m \in \mathbb{Z}.$$

c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. [3]

d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. [8]

Markscheme

a. even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = (\cos(-2x))(\cos(-4x))\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

b. consider the case $n = 1$

$\frac{\sin 4x}{2\sin 2x} = \frac{2\sin 2x \cos 2x}{2\sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x)\ldots(\cos 2^k x) = \frac{\sin 2^{k+1}x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1}x)$ ***(M1)***

$= \frac{\sin 2^{k+1}x}{2^k \sin 2x} \cos 2^{k+1}x$ ***A1***

$= \frac{2 \sin 2^{k+1}x \cos 2^{k+1}x}{2^{k+1} \sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2}x}{2^{k+1} \sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

c. attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$f_n'(x) = \frac{(2^n \sin 2x)(2^{n+1} \cos 2^{n+1}x) - (\sin 2^{n+1}x)(2^{n+1} \cos 2x)}{(2^n \sin 2x)^2}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

d. $f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin \frac{\pi}{2}\right)\left(2^{n+1} \cos 2^{n+1}\frac{\pi}{4}\right) - \left(\sin 2^{n+1}\frac{\pi}{4}\right)\left(2^{n+1} \cos \frac{\pi}{2}\right)}{\left(2^n \sin \frac{\pi}{2}\right)^2}$ ***(M1)(A1)***

$f_n'\left(\frac{\pi}{4}\right) = \frac{(2^n)\left(2^{n+1} \cos 2^{n+1}\frac{\pi}{4}\right)}{(2^n)^2}$ ***(A1)***

$= 2 \cos 2^{n+1}\frac{\pi}{4}$ $(= 2 \cos 2^{n-1}\pi)$ ***A1***

$f_n'\left(\frac{\pi}{4}\right) = 2$ ***A1***

$f_n\left(\frac{\pi}{4}\right) = 0$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$y = 2\left(x - \frac{\pi}{4}\right)$ ***M1A1***

$4x - 2y - \pi = 0$ ***AG***

[8 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Consider the functions $f(x) = \tan x,\ 0 \le\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1$.

a. Find an expression for $g \circ f(x)$, stating its domain. [2]

b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

c. Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

Markscheme

a. $g \circ f(x) = \frac{\tan x + 1}{\tan x - 1}$ ***A1***

$x \ne \frac{\pi}{4},\ 0 \le x < \frac{\pi}{2}$ ***A1***

[2 marks]

b. $\frac{\tan x + 1}{\tan x - 1} = \frac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ ***M1A1***

$= \frac{\sin x + \cos x}{\sin x - \cos x}$ ***AG***

[2 marks]

c. **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ ***M1(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2\sin x\cos x - \cos^2 x - \sin^2 x) - (2\sin x\cos x + \cos^2 x + \sin^2 x)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \frac{-2}{1 - \sin 2x}$

Substitute $\frac{\pi}{6}$ into any formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{-2}{1 - \sin\frac{\pi}{3}}$

$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$ ***A1***

$= \frac{-4}{2 - \sqrt{3}}$

$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ ***M1***

$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ ***A1***

METHOD 2

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$ ***M1A1***

$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$ ***A1***

$= \frac{-2\sec^2\frac{\pi}{6}}{\left(\tan\frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$ ***M1***

Note: Award ***M1*** for substitution $\frac{\pi}{6}$.

$\frac{-8}{\left(1-\sqrt{3}\right)^2} = \frac{-8}{\left(4-2\sqrt{3}\right)}\frac{\left(4+2\sqrt{3}\right)}{\left(4+2\sqrt{3}\right)} = -8-4\sqrt{3}$ ***M1A1***

[6 marks]

d. Area $\left|\int_0^{\frac{\pi}{6}} \frac{\sin x+\cos x}{\sin x-\cos x}\mathrm{d}x\right|$ ***M1***

$= \left|\left[\ln\left|\sin x - \cos x\right|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln\left|\sin 0 - \cos 0\right|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Find $\int \arcsin x\,\mathrm{d}x$

Markscheme

attempt at integration by parts with $u = \arcsin x$ and $v' = 1$ ***M1***

$\int \arcsin x\,\mathrm{d}x = x\arcsin x - \int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for $x\arcsin x$ and ***A1*** for $-\int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$.

solving $\int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ by substitution with $u = 1-x^2$ or inspection ***(M1)***

$\int \arcsin x\mathrm{d}x = x\arcsin x + \sqrt{1-x^2} + c$ ***A1***

[5 marks]

Examiners report

[N/A]

Show that $\int_0^{\frac{\pi}{6}} x \sin 2x \mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

Markscheme

Using integration by parts ***(M1)***

$u = x,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x$ and $v = -\frac{1}{2}\cos 2x$ ***(A1)***

$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(-\frac{1}{2}\cos 2x\right)\mathrm{d}x$ ***A1***

$= \left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$ ***A1***

Note: Award the ***A1A1*** above if the limits are not included.

$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} = -\frac{\pi}{24}$ ***A1***

$\left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{8}$ ***A1***

$\int_0^{\frac{\pi}{6}} x \sin 2x \mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$ ***AG*** ***N0***

Note: Allow ***FT*** on the last two ***A1*** marks if the expressions are the negative of the correct ones.

[6 marks]

Examiners report

This question was reasonably well done, with few candidates making the inappropriate choice of *u* and $\frac{\mathrm{d}v}{\mathrm{d}x}$. The main source of a loss of marks was in finding *v* by integration. A few candidates used the double angle formula for sine, with poor results.

A function f is defined by $f(x) = \frac{3x-2}{2x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{2}$.

a. Find an expression for $f^{-1}(x)$. [4]

b. Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. [2]

c. Hence, write down $\int \frac{3x-2}{2x-1}\mathrm{d}x$. [1]

Markscheme

a. $f : x \to y = \frac{3x-2}{2x-1}$ $\quad f^{-1} : y \to x$

$y = \frac{3x-2}{2x-1} \Rightarrow 3x - 2 = 2xy - y$ ***M1***

$\Rightarrow 3x - 2xy = -y + 2$ ***M1***

$x(3-2y)=2-y$

$x=\frac{2-y}{3-2y}$ ***A1***

$\left(f^{-1}(y)=\frac{2-y}{3-2y}\right)$

$f^{-1}(x)=\frac{2-x}{3-2x}\quad\left(x\neq\frac{3}{2}\right)$ ***A1***

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

b. $\frac{3x-2}{2x-1}=A+\frac{B}{2x-1}\Rightarrow 3x-2=A(2x-1)+B$

equating coefficients $3=2A$ and $-2=-A+B$ ***(M1)***

$A=\frac{3}{2}$ and $B=-\frac{1}{2}$ ***A1***

Note: Could also be done by division or substitution of values.

[2 marks]

c. $\int f(x)\mathrm{d}x=\frac{3}{2}x-\frac{1}{4}\ln|2x-1|+c$ ***A1***

Note: accept equivalent e.g. $\ln|4x-2|$

[1 mark]

Total [7 marks]

Examiners report

a. Well done. Only a few candidates confused inverse with derivative or reciprocal.

b. Not enough had the method of polynomial division.

c. Reasonable if they had an answer to (b) (follow through was given) usual mistakes with not allowing for the derivative of the bracket.

The function f is defined as $f(x)=ax^2+bx+c$ where $a,\ b,\ c\in\mathbb{R}$.

Hayley conjectures that $\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{f'(x_2)+f'(x_1)}{2},\ x1\neq x2.$

Show that Hayley's conjecture is correct.

Markscheme

$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{ax_2^2+bx_2+c-(ax_1^2+bx_1+c)}{x_2-x_1}$ ***(M1)***

$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1}$ ***A1***

$= \frac{a(x_2 - x_1)(x_2 + x_1) + b(x_2 - x_1)}{x_2 - x_1}$ ***(A1)***

$= a(x_2 + x_1) + b \quad (x_1 \neq x_2)$ ***A1***

$\frac{f'(x_2) + f'(x_1)}{2} = \frac{(2ax_2 + b) + (2ax_1 + b)}{2}$ ***M1***

$= \frac{2a(x_2 + x_1) + 2b}{2}$

$= a(x_2 + x_1) + b$ ***A1***

so Hayley's conjecture is correct ***AG***

[6 marks]

Examiners report

This was generally answered very well. A small minority attempted to 'prove' the result by substituting specific values into the identity and thus gained little or no credit. Some started by assuming the result to be correct, then manipulated both sides until they derived an obvious identity. Reluctantly, they gained credit for this, though such an approach should be discouraged.

Find the x-coordinates of all the points on the curve $y = 2x^4 + 6x^3 + \frac{7}{2}x^2 - 5x + \frac{3}{2}$ at which the tangent to the curve is parallel to the tangent at $(-1,\ 6)$.

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3 + 18x^2 + 7x - 5$ ***A1***

when $x = -1,\ \frac{\mathrm{d}y}{\mathrm{d}x} = -2$ ***A1***

$8x^3 + 18x^2 + 7x - 5 = -2$ ***M1***

$8x^3 + 18x^2 + 7x - 3 = 0$

$(x + 1)$ is a factor ***A1***

$8x^3 + 18x^2 + 7x - 3 = (x + 1)(8x^2 + 10x - 3)$ ***(M1)***

Note: ***M1*** is for attempting to find the quadratic factor.

$(x + 1)(4x - 1)(2x + 3) = 0$

$(x = -1),\ x = 0.25,\ x = -1.5$ ***(M1)A1***

Note: ***M1*** is for an attempt to solve their quadratic factor.

[7 marks]

Examiners report

The first half of the question was accessible to all the candidates. Some though saw the word 'tangent' and lost time calculating the equation of this. It was a pity that so many failed to spot that $x + 1$ was a factor of the cubic and so did not make much progress with the final part of this question.

Consider the curve $y=\frac{1}{1-x}+\frac{4}{x-4}$.

Find the *x*-coordinates of the points on the curve where the gradient is zero.

Markscheme

valid attempt to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(1-x)^2}-\frac{4}{(x-4)^2}$ ***A1A1***

attempt to solve $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***M1***

$x=2,\ x=-2$ ***A1A1***

[6 marks]

Examiners report

[N/A]

Use the substitution $x=a\sec\theta$ to show that $\int_{a\sqrt{2}}^{2a}\frac{\mathrm{d}x}{x^3\sqrt{x^2-a^2}}=\frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$.

Markscheme

$x=a\sec\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=a\sec\theta\tan\theta$ ***(A1)***

new limits:

$x=a\sqrt{2}\Rightarrow\theta=\frac{\pi}{4}$ and $x=2a\Rightarrow\theta=\frac{\pi}{3}$ ***(A1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta-a^2}}\mathrm{d}\theta$ ***M1***

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos^2\theta}{a^3}\mathrm{d}\theta$ ***A1***

using $\cos^2\theta=\frac{1}{2}(\cos 2\theta+1)$ ***M1***

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent ***A1***

$=\frac{1}{4a^3}\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}-1-\frac{\pi}{2}\right)$ or equivalent ***A1***

$=\frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$ ***AG***

[7 marks]

Examiners report

[N/A]

---

a. Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2 x}{\sqrt[3]{\tan x}}\mathrm{d}x$. [6]

b. Find $\int\tan^3 x\mathrm{d}x$. [3]

Markscheme

a. **EITHER**

let $u = \tan x;\ \mathrm{d}u = \sec^2 x \mathrm{d}x$ ***(M1)***

consideration of change of limits ***(M1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u^{\frac{1}{3}}} \mathrm{d}u$ ***(A1)***

Note: Do not penalize lack of limits.

$= \left[\frac{3u^{\frac{2}{3}}}{2}\right]_1^{\sqrt{3}}$ ***A1***

$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2}\right)$ ***A1A1*** ***N0***

OR

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} \mathrm{d}x = \left[\frac{3(\tan x)^{\frac{2}{3}}}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ ***M2A2***

$= \frac{3 \times \sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3} - 3}{2}\right)$ ***A1A1*** ***N0***

[6 marks]

b. $\int \tan^3 x \mathrm{d}x = \int \tan x(\sec^2 x - 1)\mathrm{d}x$ ***M1***

$= \int (\tan x \times \sec^2 x - \tan x)\mathrm{d}x$

$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C$ ***A1A1***

Note: Do not penalize the absence of absolute value or *C*.

[3 marks]

Examiners report

a. Quite a variety of methods were successfully employed to solve part (a).

b. Many candidates did not attempt part (b).

Find the value of $\int_0^1 t \ln(t+1)\mathrm{d}t$.

Markscheme

EITHER

attempt at integration by substitution ***(M1)***

using $u = t + 1$, $\mathrm{d}u = \mathrm{d}t$, the integral becomes

$\int_1^2 (u-1) \ln u \mathrm{d}u$ ***A1***

then using integration by parts ***M1***

$\int_1^2 (u-1) \ln u \mathrm{d}u = \left[\left(\frac{u^2}{2} - u\right) \ln u\right]_1^2 - \int_1^2 \left(\frac{u^2}{2} - u\right) \times \frac{1}{u} \mathrm{d}u$ ***A1***

$= -\left[\frac{u^2}{4} - u\right]_1^2$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

OR

attempt to integrate by parts ***(M1)***

correct choice of variables to integrate and differentiate ***M1***

$\int_0^1 t\ln(t+1)\mathrm{d}t = \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \int_0^1 \frac{t^2}{2} \times \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\int_0^1 t - 1 + \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right]_0^1$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

[6 marks]

Examiners report

Again very few candidates gained full marks on this question. The most common approach was to begin by integrating by parts, which was done correctly, but very few candidates then knew how to integrate $\frac{t^2}{t+1}$. Those who began with a substitution often made more progress. Again a number of candidates were let down by their inability to simplify appropriately.

Let $y = \arccos\left(\frac{x}{2}\right)$

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Find $\int_0^1 \arccos\left(\frac{x}{2}\right)\mathrm{d}x$. [7]

Markscheme

a. $y = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \left(= -\frac{1}{\sqrt{4-x^2}}\right)$ ***M1A1***

Note: ***M1*** is for use of the chain rule.

[2 marks]

b. attempt at integration by parts ***M1***

$u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{4-x^2}}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$ ***(A1)***

$\int_0^1 \arccos\left(\frac{x}{2}\right)\mathrm{d}x = \left[x \arccos\left(\frac{x}{2}\right)\right]_0^1 + \int_0^1 \frac{1}{\sqrt{4-x^2}}dx$ ***A1***

using integration by substitution or inspection ***(M1)***

$\left[x \arccos\left(\frac{x}{2}\right)\right]_0^1 + \left[-\left(4-x^2\right)^{\frac{1}{2}}\right]_0^1$ ***A1***

Note: Award ***A1*** for $-\left(4-x^2\right)^{\frac{1}{2}}$ or equivalent.

Note: Condone lack of limits to this point.

attempt to substitute limits into their integral ***M1***

$= \frac{\pi}{3} - \sqrt{3} + 2$ ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the function $f(x) = \frac{1}{x^2+3x+2},\ x \in \mathbb{R},\ x \neq -2,\ x \neq -1.$

a.i. Express $x^2 + 3x + 2$ in the form $(x+h)^2 + k$. [1]

a.ii. Factorize $x^2 + 3x + 2$. [1]

b. Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum. [5]

c. Show that $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x^2+3x+2}$. [1]

d. Hence find the value of p if $\int_0^1 f(x)\mathrm{d}x = \ln(p)$. [4]

e. Sketch the graph of $y = f(|x|)$. [2]

f. Determine the area of the region enclosed between the graph of $y = f(|x|)$, the x-axis and the lines with equations $x = -1$ and $x = 1$. [3]

Markscheme

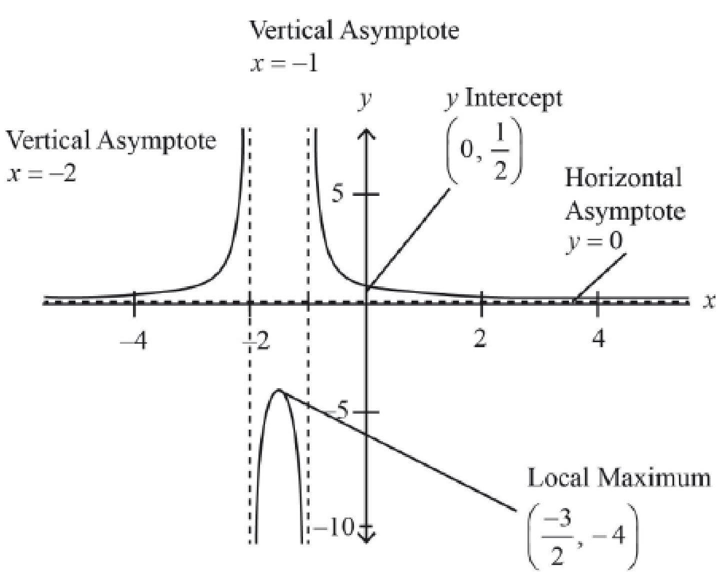
a.i. $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

[1 mark]

a.ii. $x^2 + 3x + 2 = (x+2)(x+1)$ ***A1***

[1 mark]

b.



A1 for the shape

A1 for the equation $y = 0$

A1 for asymptotes $x = -2$ and $x = -1$

A1 for coordinates $\left(-\frac{3}{2},\ -4\right)$

A1 y-intercept $\left(0,\ \frac{1}{2}\right)$

[5 marks]

c. $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)(x+2)}$ ***M1***

$= \frac{1}{x^2+3x+2}$ ***AG***

[1 mark]

d. $\int\limits_0^1 \frac{1}{x+1} - \frac{1}{x+2} \mathrm{d}x$

$= [\ln(x+1) - \ln(x+2)]_0^1$ ***A1***

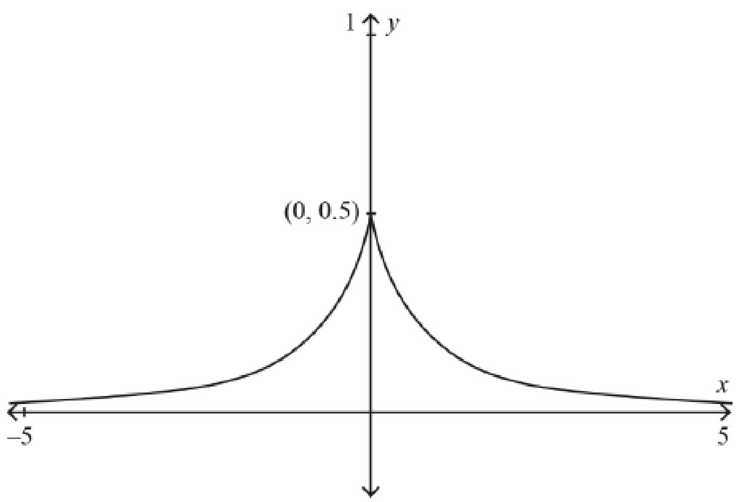
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore p = \frac{4}{3}$

[4 marks]

e.



symmetry about the y-axis ***M1***

correct shape ***A1***

Note: Allow ***FT*** from part (b).

[2 marks]

f. $2\int_0^1 f(x)\mathrm{d}x$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

Note: Do not award ***FT*** from part (e).

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]

A particle moves in a straight line such that at time t seconds $(t \geqslant 0)$, its velocity v, in ms^{-1}, is given by $v = 10t\mathrm{e}^{-2t}$. Find the exact distance travelled by the particle in the first half-second.

Markscheme

$$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t$$

attempt at integration by parts ***M1***

$$= \left[-5t\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -5\mathrm{e}^{-2t}\mathrm{d}t \quad \textbf{\textit{A1}}$$

$$= \left[-5t\mathrm{e}^{-2t} - \frac{5}{2}\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}} \quad \textbf{\textit{(A1)}}$$

Note: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t \quad \textbf{\textit{(M1)}}$$

$$= -5\mathrm{e}^{-1} + \frac{5}{2} \left(= \frac{-5}{\mathrm{e}} + \frac{5}{2}\right) \left(= \frac{5\mathrm{e}-10}{2\mathrm{e}}\right) \quad \textbf{\textit{A1}}$$

[5 marks]

Examiners report

[N/A]

(a) Show that $\frac{3}{x+1} + \frac{2}{x+3} = \frac{5x+11}{x^2+4x+3}$.

(b) Hence find the value of k such that $\int_0^2 \frac{5x+11}{x^2+4x+3}\mathrm{d}x = \ln k$.

Markscheme

(a) $\frac{3}{x+1} + \frac{2}{x+3} = \frac{3(x+3)+2(x+1)}{(x+1)(x+3)}$ ***M1***

$= \frac{3x+9+2x+2}{x^2+4x+3}$ ***A1***

$= \frac{5x+11}{x^2+4x+3}$ ***AG***

(b) $\int_0^2 \frac{5x+11}{x^2+4x+3}\mathrm{d}x = \int_0^2 \left(\frac{3}{x+1} + \frac{2}{x+3}\right)\mathrm{d}x$ ***M1***

$= [3\ln(x+1) + 2\ln(x+3)]_0^2$ ***A1***

$= 3\ln 3 + 2\ln 5 - 3\ln 1 - 2\ln 3 \quad (= 3\ln 3 + 2\ln 5 - 2\ln 3)$ ***A1***

$= \ln 3 + 2\ln 5$

$= \ln 75 \quad (k = 75)$ ***A1***

[6 marks]

Examiners report

Many students did not 'Show' enough in a) in order to be convincing. The need for the steps of the simplification to be shown was not clear. Too many did not link a) to b) and seemed to not be aware of the Command Term 'hence' and its implication for marking (no marks will be awarded to alternative methods). The simplifications of the log expressions were done poorly by many and the fact that $3^3 = 9$ was noted by too many. There were very few elegant solutions to this question.

---

a. Use the substitution $u = x^{\frac{1}{2}}$ to find $\int \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}$. [4]

b. Hence find the value of $\frac{1}{2}\int\limits_1^9 \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}$, expressing your answer in the form arctan q, where $q \in \mathbb{Q}$. [3]

Markscheme

a. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$ (accept $\mathrm{d}u = \frac{1}{2}x^{-\frac{1}{2}}\mathrm{d}x$ or equivalent) ***A1***

substitution, leading to an integrand in terms of u ***M1***

$\int \frac{2u\mathrm{d}u}{u^3+u}$ or equivalent ***A1***

= 2 arctan $\left(\sqrt{x}\right)(+c)$ ***A1***

[4 marks]

b.

$\frac{1}{2}\int\limits_1^9 \frac{\mathrm{d}x}{x^{\frac{3}{2}} + x^{\frac{1}{2}}}$ = arctan3 – arctan1 ***A1***

tan(arctan3 – arctan1) = $\frac{3-1}{1+3\times 1}$ ***(M1)***

tan(arctan3 – arctan1) = $\frac{1}{2}$

arctan3 – arctan1 = arctan$\frac{1}{2}$ ***A1***

[3 marks]

Examiners report

[N/A]

b. [N/A]

---

a. Find all values of x for $0.1 \leqslant x \leqslant 1$ such that $\sin(\pi x^{-1}) = 0$. [2]

b. Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2} \sin(\pi x^{-1}) \mathrm{d}x$, showing that it takes different integer values when n is even and when n is odd. [3]

c. Evaluate $\int_{0.1}^{1} \left|\pi x^{-2} \sin(\pi x^{-1})\right| \mathrm{d}x$. [2]

Markscheme

a. $\sin(\pi x^{-1}) = 0 \ \frac{\pi}{x} = \pi,\ 2\pi(\ldots)$ ***(A1)***

$x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ ***A1***

[2 marks]

b. $\left[\cos(\pi x^{-1})\right]_{\frac{1}{n+1}}^{\frac{1}{n}}$ ***M1***

$= \cos(\pi n) - \cos(\pi(n+1))$ ***A1***

= 2 when n is even and = –2 when n is odd ***A1***

[3 marks]

c. $\int_{0.1}^{1} \left|\pi x^{-2} \sin(\pi x^{-1})\right| \mathrm{d}x = 2 + 2 + \ldots + 2 = 18$ ***(M1)A1***

[2 marks]

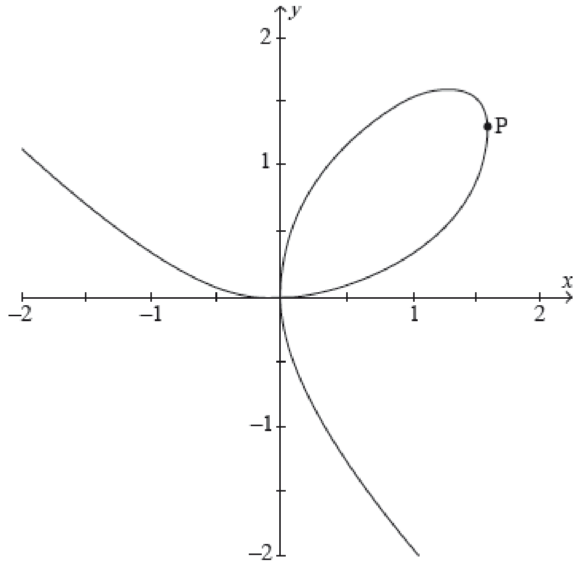
Examiners report

a. There were a pleasing number of candidates who answered part (a) correctly. Fewer were successful with part (b). It was expected by this stage of the paper that candidates would be able to just write down the value of the integral rather than use substitution to evaluate it.

b. There were a pleasing number of candidates who answered part (a) correctly. Fewer were successful with part (b). It was expected by this stage of the paper that candidates would be able to just write down the value of the integral rather than use substitution to evaluate it.

c. There were disappointingly few correct answers to part (c) with candidates not realising that it was necessary to combine the previous two parts in order to write down the answer.

The folium of Descartes is a curve defined by the equation $x^3 + y^3 - 3xy = 0$, shown in the following diagram.



Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the y-axis.

Markscheme

$x^3 + y^3 - 3xy = 0$

$3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$ ***M1A1***

Note: Differentiation wrt y is also acceptable.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y-3x^2}{3y^2-3x} \left(= \frac{y-x^2}{y^2-x}\right)$ ***(A1)***

Note: All following marks may be awarded if the denominator is correct, but the numerator incorrect.

$y^2 - x = 0$ ***M1***

EITHER

$x = y^2$

$y^6 + y^3 - 3y^3 = 0$ ***M1A1***

$y^6 - 2y^3 = 0$

$y^3(y^3 - 2) = 0$

$(y \neq 0) \therefore y = \sqrt[3]{2}$ ***A1***

$x = \left(\sqrt[3]{2}\right)^2 \left(= \sqrt[3]{4}\right)$ ***A1***

OR

$x^3 + xy - 3xy = 0$ ***M1***

$x(x^2 - 2y) = 0$

$x \neq 0 \Rightarrow y = \frac{x^2}{2}$ ***A1***

$y^2 = \frac{x^4}{4}$

$x = \frac{x^4}{4}$

$x(x^3 - 4) = 0$

$(x \neq 0) \therefore x = \sqrt[3]{4}$ ***A1***

$y = \frac{\left(\sqrt[3]{4}\right)^2}{2} = \sqrt[3]{2}$ ***A1***

[8 marks]

Examiners report

[N/A]

A body is moving in a straight line. When it is s metres from a fixed point O on the line its velocity, v, is given by $v = -\frac{1}{s^2},\ s > 0$.

Find the acceleration of the body when it is 50 cm from O.

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}s} = 2s^{-3}$ ***M1A1***

Note: Award ***M1*** for $2s^{-3}$ and ***A1*** for the whole expression.

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$ ***(M1)***
$a = -\frac{1}{s^2} \times \frac{2}{s^3} \left(= -\frac{2}{s^5}\right)$ ***(A1)***
when $s = \frac{1}{2},\ a = -\frac{2}{(0.5)^5}\ (= -64)\ (\mathrm{ms}^{-2})$ ***M1A1***

Note: ***M1*** is for the substitution of 0.5 into their equation for acceleration.
Award ***M1A0*** if $s = 50$ is substituted into the correct equation.

[6 marks]

Examiners report

[N/A]

Consider the curve $y = x\mathrm{e}^x$ and the line $y = kx,\ k \in \mathbb{R}$.

(a) Let $k = 0$.

(i) Show that the curve and the line intersect once.

(ii) Find the angle between the tangent to the curve and the line at the point of intersection.

(b) Let $k = 1$. Show that the line is a tangent to the curve.

(c) (i) Find the values of k for which the curve $y = x\mathrm{e}^x$ and the line $y = kx$ meet in two distinct points.

(ii) Write down the coordinates of the points of intersection.

(iii) Write down an integral representing the area of the region A enclosed by the curve and the line.

(iv) **Hence**, given that $0 < k < 1$, show that $A < 1$.

Markscheme

(a) (i) $xe^x = 0 \Rightarrow x = 0$ ***A1***

so, they intersect only once at (0, 0)

(ii) $y' = e^x + xe^x = (1+x)e^x$ ***M1A1***

$y'(0) = 1$ ***A1***

$\theta = \arctan 1 = \frac{\pi}{4}$ $(\theta = 45^\circ)$ ***A1***

[5 marks]

(b) when $k = 1,\ y = x$

$xe^x = x \Rightarrow x(e^x - 1) = 0$ ***M1***

$\Rightarrow x = 0$ ***A1***

$y'(0) = 1$ which equals the gradient of the line $y = x$ ***R1***

so, the line is tangent to the curve at origin ***AG***

Note: Award full credit to candidates who note that the equation $x(e^x - 1) = 0$ has a double root $x = 0$ so $y = x$ is a tangent.

[3 marks]

(c) (i) $xe^x = kx \Rightarrow x(e^x - k) = 0$ ***M1***

$\Rightarrow x = 0$ or $x = \ln k$ ***A1***

$k > 0$ and $k \neq 1$ ***A1***

(ii) (0, 0) and $(\ln k,\ k\ln k)$ ***A1A1***

(iii) $A = \left|\int_0^{\ln k} kx - xe^x \mathrm{d}x\right|$ ***M1A1***

Note: Do not penalize the omission of absolute value.

(iv) attempt at integration by parts to find $\int xe^x \mathrm{d}x$ ***M1***

$\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x = e^x(x-1)$ ***A1***

as $0 < k < 1 \Rightarrow \ln k < 0$ ***R1***

$A = \int_{\ln k}^{0} kx - xe^x \mathrm{d}x = \left[\frac{k}{2}x^2 - (x-1)e^x\right]_{\ln k}^{0}$ ***A1***

$= 1 - \left(\frac{k}{2}(\ln k)^2 - (\ln k - 1)k\right)$ ***A1***

$= 1 - \frac{k}{2}\left((\ln k)^2 - 2\ln k + 2\right)$

$= 1 - \frac{k}{2}\left((\ln k - 1)^2 + 1\right)$ ***M1A1***

since $\frac{k}{2}\left((\ln k - 1)^2 + 1\right) > 0$ ***R1***

$A < 1$ ***AG***

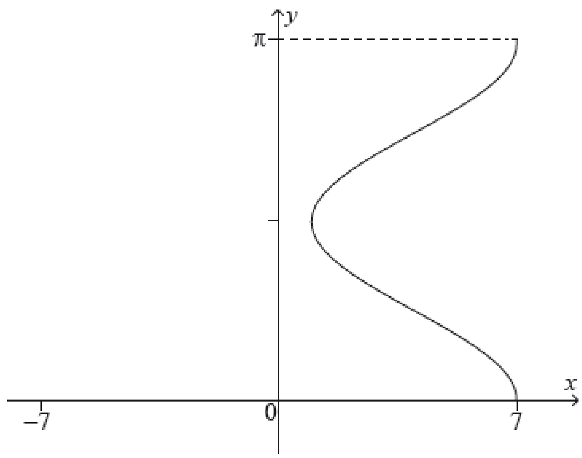
[15 marks]

Total [23 marks]

Examiners report

Many candidates solved (a) and (b) correctly but in (c), many failed to realise that the equation $xe^x = kx$ has two roots under certain conditions and that the point of the question was to identify those conditions. Most candidates made a reasonable attempt to write down the appropriate integral in (c)(iii) with the modulus signs and limits often omitted but no correct solution has yet been seen to (c)(iv).

The following graph shows the relation $x = 3\cos 2y + 4,\ 0 \leqslant y \leqslant \pi$.



The curve is rotated 360° about the y-axis to form a volume of revolution.

A container with this shape is made with a solid base of diameter 14 cm . The container is filled with water at a rate of $2\ \mathrm{cm}^3\ \mathrm{min}^{-1}$. At time t minutes, the water depth is $h\ \mathrm{cm},\ 0 \leqslant h \leqslant \pi$ and the volume of water in the container is $V\ \mathrm{cm}^3$.

a. Calculate the value of the volume generated. [8]

b. (i) Given that $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi(3\cos 2h + 4)^2$, find an expression for $\frac{\mathrm{d}h}{\mathrm{d}t}$. [4]

(ii) Find the value of $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = \frac{\pi}{4}$.

c. (i) Find $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$. [7]

(ii) Find the values of h for which $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = 0$.

(iii) By making specific reference to the shape of the container, interpret $\frac{\mathrm{d}h}{\mathrm{d}t}$ at the values of h found in part (c)(ii).

Markscheme

a. use of $\pi\int_a^b x^2\mathrm{d}y$ ***(M1)***

Note: Condone any or missing limits.

$V = \pi\int_0^\pi (3\cos 2y + 4)^2\mathrm{d}y$ ***(A1)***

$= \pi\int_0^\pi (9\cos^2 2y + 24\cos 2y + 16)\mathrm{d}y$ ***A1***

$9\cos^2 2y = \frac{9}{2}(1 + \cos 4y)$ ***(M1)***

$= \pi\left[\frac{9y}{2} + \frac{9}{8}\sin 4y + 12\sin 2y + 16y\right]_0^\pi$ ***M1A1***

$= \pi\left(\frac{9\pi}{2} + 16\pi\right)$ ***(A1)***

$= \frac{41\pi^2}{2}\ (\mathrm{cm}^3)$ ***A1***

Note: If the coefficient "π" is absent, or *eg*, "2π" is used, only ***M*** marks are available.

[8 marks]

b. (i) attempting to use $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ with $\frac{\mathrm{d}V}{\mathrm{d}t} = 2$ ***M1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{\pi(3\cos 2h+4)^2}$ ***A1***

(ii) substituting $h = \frac{\pi}{4}$ into $\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{8\pi}\ (\mathrm{cm\,min}^{-1})$ ***A1***

Note: Do not allow FT marks for (b)(ii).

[4 marks]

c. (i) $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right) = \frac{\mathrm{d}h}{\mathrm{d}t} \times \frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$ ***(M1)***

$= \frac{2}{\pi(3\cos 2h+4)^2} \times \frac{24\sin 2h}{\pi(3\cos 2h+4)^3}$ ***M1A1***

Note: Award ***M1*** for attempting to find $\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$.

$= \frac{48\sin 2h}{\pi^2(3\cos 2h+4)^5}$ ***A1***

(ii) $\sin 2h = 0 \Rightarrow h = 0,\ \frac{\pi}{2},\ \pi$ ***A1***

Note: Award ***A1*** for $\sin 2h = 0 \Rightarrow h = 0,\ \frac{\pi}{2},\ \pi$ from an incorrect $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$.

(iii) **METHOD 1**

$\frac{\mathrm{d}h}{\mathrm{d}t}$ is a minimum at $h = 0,\ \pi$ and the container is widest at these values ***R1***

$\frac{\mathrm{d}h}{\mathrm{d}t}$ is a maximum at $h = \frac{\pi}{2}$ and the container is narrowest at this value ***R1***

[7 marks]

Examiners report

a. Part (a) was often answered well, though for some reason a minority tended to use the incorrect $\pi\int(3\cos 2y)^2\mathrm{d}y$ and gained few marks thereafter. Incorrect limits were sometimes seen, which led to only method marks being available. A pleasing number were able to deal with the integration of $\cos^2 2y$ through the use of the correct identity.

b. Part (b) was well answered and did not pose too many problems.

c. Correct answers to part (c) were rarely seen. Only the very best candidates appreciated the correct use of the chain rule when trying to determine an expression for $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$.

The curve C has equation $y = \frac{1}{8}(9 + 8x^2 - x^4)$.

a. Find the coordinates of the points on C at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [4]

b. The tangent to C at the point P(1, 2) cuts the x-axis at the point T. Determine the coordinates of T. [4]

c. The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN. [7]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{1}{2}x^3$ ***A1***

$x\left(2 - \frac{1}{2}x^2\right) = 0$

$x = 0,\ \pm 2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $\left(0, \frac{9}{8}\right),\ \left(-2, \frac{25}{8}\right),\ \left(2, \frac{25}{8}\right)$ ***A1A1A1***

Note: Award ***A2*** for all three x-values correct with errors/omissions in y-values.

[4 marks]

b. at x =1, gradient of tangent $= \frac{3}{2}$ ***(A1)***

Note: In the following, allow ***FT*** on incorrect gradient.

equation of tangent is $y - 2 = \frac{3}{2}(x-1)$ $\left(y = \frac{3}{2}x + \frac{1}{2}\right)$ ***(A1)***

meets x-axis when y = 0 , $-2 = \frac{3}{2}(x-1)$ ***(M1)***

$x = -\frac{1}{3}$

coordinates of T are $\left(-\frac{1}{3}, 0\right)$ ***A1***

[4 marks]

c. gradient of normal $= -\frac{2}{3}$ ***(A1)***

equation of normal is $y - 2 = -\frac{2}{3}(x-1)$ $\left(y = -\frac{2}{3}x + \frac{8}{3}\right)$ ***(M1)***

at x = 0 , $y = \frac{8}{3}$ ***A1***

Note: In the following, allow FT on incorrect coordinates of T and N.

lengths of PN $= \sqrt{\frac{13}{9}}$, PT $= \sqrt{\frac{52}{9}}$ ***A1A1***

area of triangle PTN $= \frac{1}{2} \times \sqrt{\frac{13}{9}} \times \sqrt{\frac{52}{9}}$ ***M1***

$= \frac{13}{9}$ (or equivalent *e.g.* $\frac{\sqrt{676}}{18}$) ***A1***

[7 marks]

Examiners report

a. The whole of this question seemed to prove accessible to a high proportion of candidates.

(a) was well answered by most, although a number of candidates gave only the x-values of the points or omitted the value at 0.
(b) was successfully solved by the majority of candidates, who also found the correct equation of the normal in (c).
The last section proved more difficult for many candidates, the most common error being to use the wrong perpendicular sides. There were a number of different approaches here all of which were potentially correct but errors abounded.

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The last section proved more difficult for many candidates, the most common error being to use the wrong perpendicular sides. There were a number of different approaches here all of which were potentially correct but errors abounded.

The function f is defined by

$$f(x) = \begin{cases} 2x - 1, & x \leqslant 2 \\ ax^2 + bx - 5, & 2 < x < 3 \end{cases}$$

where a , $b \in \mathbb{R}$.

a. Given that f and its derivative, f' , are continuous for all values in the domain of f, find the values of a and b . [6]

b. Show that f is a one-to-one function. [3]

c. Obtain expressions for the inverse function f^{-1} and state their domains. [5]

Markscheme

a. f continuous $\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ ***M1***

$4a + 2b = 8$ ***A1***

$f'(x) = \begin{cases} 2, & x < 2 \\ 2ax + b, & 2 < x < 3 \end{cases}$ ***A1***

f' continuous $\Rightarrow \lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$

$4a + b = 2$ ***A1***

solve simultaneously ***M1***

to obtain $a = -1$ and $b = 6$ ***A1***

[6 marks]

b. for $x \leqslant 2,\ f'(x) = 2 > 0$ ***A1***

for $2 < x < 3,\ f'(x) = -2x + 6 > 0$ ***A1***

since $f'(x) > 0$ for all values in the domain of f, f is increasing ***R1***

therefore one-to-one ***AG***

[3 marks]

c. $x = 2y - 1 \Rightarrow y = \frac{x+1}{2}$ ***M1***

$x = -y^2 + 6y - 5 \Rightarrow y^2 - 6y + x + 5 = 0$ ***M1***

$y = 3 \pm \sqrt{4 - x}$

therefore

$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & x \leqslant 3 \\ 3 - \sqrt{4 - x}, & 3 < x < 4 \end{cases}$ ***A1A1A1***

Note: Award ***A1*** for the first line and ***A1A1*** for the second line.

[5 marks]

Examiners report

[N/A]

a. b. [N/A]

c. [N/A]

A curve is given by the equation $y = \sin(\pi \cos x)$.

Find the coordinates of all the points on the curve for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0,\ 0 \leqslant x \leqslant \pi$.

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\cos(\pi \cos x) \times \pi \sin x$ ***M1A1***

Note: Award follow through marks below if their answer is a multiple of the correct answer.

considering either $\sin x = 0$ or $\cos(\pi \cos x) = 0$ ***(M1)***

$x = 0,\ \pi$ ***A1***

$\pi \cos x = \frac{\pi}{2},\ -\frac{\pi}{2}\ \left(\Rightarrow \cos x = \frac{1}{2}, -\frac{1}{2}\right)$ ***M1***

Note: Condone absence of $-\frac{\pi}{2}$.

$\Rightarrow x = \frac{\pi}{3},\ \frac{2\pi}{3}$

$(0,\ 0),\ \left(\frac{\pi}{3},\ 1\right),\ (\pi, 0)$ ***A1***

$\left(\frac{2\pi}{3},\ -1\right)$ ***A1***

[7 marks]

Examiners report

This was not a straight-forward differentiation and it was pleasing to see how many candidates managed to do it correctly. Having done this they found two of the solutions, often three of the solutions, successfully. The final solution was found by only a few candidates. Again candidates lost marks unnecessarily by not close reading the question and realising that they needed both coordinates of the points, not just the x-coordinates.

It is given that $\log_2 y + \log_4 x + \log_4 2x = 0$.

a. Show that $\log_{r^2} x = \frac{1}{2} \log_r x$ where $r,\ x \in \mathbb{R}^+$. [2]

b. Express y in terms of x. Give your answer in the form $y = px^q$, where *p* , *q* are constants. [5]

c. The region *R*, is bounded by the graph of the function found in part (b), the *x*-axis, and the lines $x = 1$ and $x = \alpha$ where $\alpha > 1$. The area of *R* is $\sqrt{2}$. [5]

Find the value of α.

Markscheme

a. **METHOD 1**

$\log_{r^2} x = \frac{\log_r x}{\log_r r^2} \left(= \frac{\log_r x}{2\log_r r}\right)$ ***M1A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

METHOD 2

$\log_{r^2} x = \frac{1}{\log_x r^2}$ ***M1***

$= \frac{1}{2\log_x r}$ ***A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

b. **METHOD 1**

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \log_4 2x^2 = 0$ ***M1***

$\log_2 y + \frac{1}{2}\log_2 2x^2 = 0$ ***M1***

$\log_2 y = -\frac{1}{2}\log_2 2x^2$

$\log_2 y = \log_2 \left(\frac{1}{\sqrt{2x}}\right)$ ***M1A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

METHOD 2

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \frac{1}{2}\log_2 x + \frac{1}{2}\log_2 2x = 0$ ***M1***

$\log_2 y + \log_2 x^{\frac{1}{2}} + \log_2 (2x)^{\frac{1}{2}} = 0$ ***M1***

$\log_2 \left(\sqrt{2}xy\right) = 0$ ***M1***

$\sqrt{2}xy = 1$ ***A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

c. the area of *R* is $\int_1^\alpha \frac{1}{\sqrt{2}}x^{-1}\mathrm{d}x$ ***M1***

$= \left[\frac{1}{\sqrt{2}}\ln x\right]_1^\alpha$ ***A1***

$= \frac{1}{\sqrt{2}}\ln\alpha$ ***A1***

$\frac{1}{\sqrt{2}} \ln \alpha = \sqrt{2}$ ***M1***

$\alpha = e^2$ ***A1***

Note: Only follow through from part (b) if y is in the form $y = px^q$

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Given that $y = \frac{1}{1-x}$, use mathematical induction to prove that $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$, $n \in \mathbb{Z}^+$.

Markscheme

proposition is true for $n = 1$ since $\frac{dy}{dx} = \frac{1}{(1-x)^2}$ ***M1***

$= \frac{1!}{(1-x)^2}$ ***A1***

Note: Must see the 1! for the ***A1***.

assume true for $n = k$, $k \in \mathbb{Z}^+$, *i.e.* $\frac{d^k y}{dx^k} = \frac{k!}{(1-x)^{k+1}}$ ***M1***

consider $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d\left(\frac{d^k y}{dx^k}\right)}{dx}$ ***(M1)***

$= (k+1)k!(1-x)^{-(k+1)-1}$ ***A1***

$= \frac{(k+1)!}{(1-x)^{k+2}}$ ***A1***

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers ***R1***

Note: The final ***R1*** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

Examiners report

Most candidates were awarded good marks for this question. A disappointing minority thought that the $(k+1)$th derivative was the (k)th derivative multiplied by the first derivative. Providing an acceptable final statement remains a perennial issue.

A curve is defined by the equation $8y \ln x - 2x^2 + 4y^2 = 7$. Find the equation of the tangent to the curve at the point where $x = 1$ and $y > 0$.

Markscheme

$8y \times \frac{1}{x} + 8\frac{dy}{dx} \ln x - 4x + 8y\frac{dy}{dx} = 0$ ***M1A1A1***

Note: ***M1*** for attempt at implicit differentiation. ***A1*** for differentiating $8y\ln x$, ***A1*** for differentiating the rest.

when $x = 1,\ 8y \times 0 - 2 \times 1 + 4y^2 = 7$ ***(M1)***

$y^2 = \frac{9}{4} \Rightarrow y = \frac{3}{2}$ (as $y > 0$) ***A1***

at $\left(1, \frac{3}{2}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}$ ***A1***

$y - \frac{3}{2} = -\frac{2}{3}(x - 1)$ or $y = -\frac{2}{3}x + \frac{13}{6}$ **A1**

[7 marks]

Examiners report

The implicit differentiation was generally well done. Some candidates did not realise that they needed to substitute into the original equation to find y. Others wasted a lot of time rearranging the derivative to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject, rather than simply putting in the particular values for x and y

.

Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of *k*, the gradient of the curve at the point (−1, *k*).

(b) Given that the tangent to the curve is parallel to the *x*-axis at this point, find the value of *k*.

Markscheme

(a) Attempting implicit differentiation ***M1***

$2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

EITHER

Substituting $x = -1,\ y = k$ *e.g.* $-2 + k - \frac{\mathrm{d}y}{\mathrm{d}x} + 2k\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

Attempting to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject ***M1***

OR

Attempting to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject *e.g.* $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(2x+y)}{x+2y}$ ***M1***

Substituting $x = -1,\ y = k$ into $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

THEN

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-k}{2k-1}$ ***A1*** ***N1***

(b) Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for k gives $k = 2$ ***A1***

[6 marks]

Examiners report

Part (a) was generally well answered, almost all candidates realising that implicit differentiation was involved. A few failed to differentiate the right hand side of the relationship. A surprising number of candidates made an error in part (b), even when they had scored full marks on the first part.

Consider the curve defined by the equation $x^2 + \sin y - xy = 0$.

a. Find the gradient of the tangent to the curve at the point $(\pi,\ \pi)$. [6]

b. Hence, show that $\tan\theta = \frac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at $(\pi,\ \pi)$ and the line $y = x$. [3]

Markscheme

a. attempt to differentiate implicitly ***M1***

$2x + \cos y\frac{\mathrm{d}y}{\mathrm{d}x} - y - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1A1***

Note: ***A1*** for differentiating x^2 and $\sin y$; ***A1*** for differentiating xy.

substitute x and y by π ***M1***

$2\pi - \frac{\mathrm{d}y}{\mathrm{d}x} - \pi - \pi\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\pi}{1+\pi}$ ***M1A1***

Note: ***M1*** for attempt to make $\mathrm{d}y/\mathrm{d}x$ the subject. This could be seen earlier.

[6 marks]

b. $\theta = \frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}$ (or seen the other way) ***M1***

$\tan\theta = \tan\left(\frac{\pi}{4} - \arctan\frac{\pi}{1+\pi}\right) = \frac{1-\frac{\pi}{1+\pi}}{1+\frac{\pi}{1+\pi}}$ ***M1A1***

$\tan\theta = \frac{1}{1+2\pi}$ ***AG***

[3 marks]

Examiners report

a. Part a) proved an easy 6 marks for most candidates, while the majority failed to make any headway with part b), with some attempting to find the equation of their line in the form $y = mx + c$. Only the best candidates were able to see their way through to the given answer.

b. Part a) proved an easy 6 marks for most candidates, while the majority failed to make any headway with part b), with some attempting to find the equation of their line in the form $y = mx + c$. Only the best candidates were able to see their way through to the given answer.

Consider the function f defined by $f(x) = x^2 - a^2,\ x \in \mathbb{R}$ where a is a positive constant.

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

a.i. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

$y = f(x)$;

a.ii.Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [4]

$y = \frac{1}{f(x)}$;

a.iiiShowing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

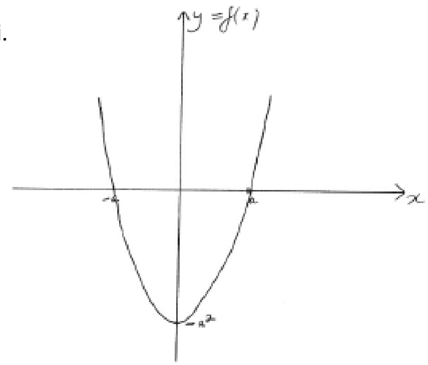
$y = \left|\frac{1}{f(x)}\right|$.

b. Find $\int f(x)\cos x\,\mathrm{d}x$. [5]

c. By finding $g'(x)$ explain why g is an increasing function. [4]

Markscheme

a.i.

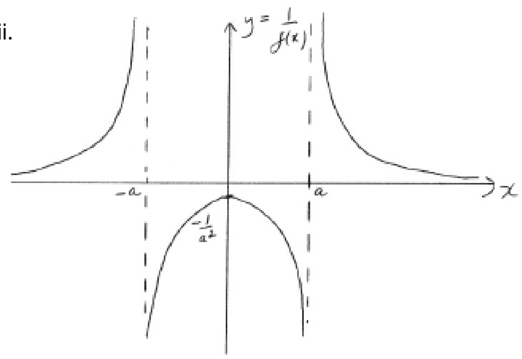


A1 for correct shape

A1 for correct x and y intercepts and minimum point

[2 marks]

a.ii.



A1 for correct shape

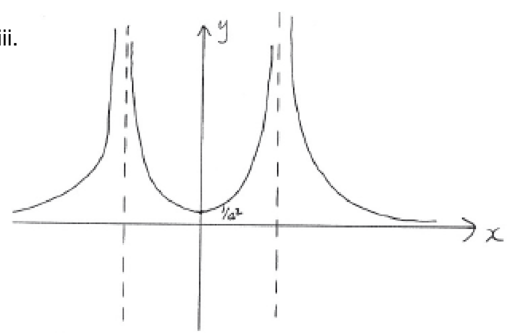
A1 for correct vertical asymptotes

A1 for correct implied horizontal asymptote

A1 for correct maximum point

[??? marks]

a.iii.



A1 for reflecting negative branch from (ii) in the x-axis

A1 for correctly labelled minimum point

[2 marks]

b. **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2) \cos x \mathrm{d}x = (x^2 - a^2) \sin x - \int 2x \sin x \mathrm{d}x$ ***A1A1***

$= (x^2 - a^2) \sin x - 2 \left[-x \cos x + \int \cos x \mathrm{d}x \right]$ ***A1***

$= (x^2 - a^2) \sin x + 2x \cos -2 \sin x + c$ ***A1***

OR

$\int (x^2 - a^2) \cos x \mathrm{d}x = \int x^2 \cos x \mathrm{d}x - \int a^2 \cos x \mathrm{d}x$

attempt at integration by parts ***(M1)***

$\int x^2 \cos x \mathrm{d}x = x^2 \sin x - \int 2x \sin x \mathrm{d}x$ ***A1A1***

$= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \mathrm{d}x \right]$ ***A1***

$= x^2 \sin x + 2x \cos x - 2 \sin x$

$- \int a^2 \cos x \mathrm{d}x = -a^2 \sin x$

$\int (x^2 - a^2) \cos x \mathrm{d}x = (x^2 - a^2) \sin x + 2x \cos x - 2 \sin x + c$ ***A1***

[5 marks]

c. $g(x) = x(x^2 - a^2)^{\frac{1}{2}}$

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + \frac{1}{2} x (x^2 - a^2)^{-\frac{1}{2}} (2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + x^2 (x^2 - a^2)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
a.iii. [N/A]
b. [N/A]
c. [N/A]

In triangle ABC, $\mathrm{BC} = \sqrt{3}$ cm, $\mathrm{A\hat{B}C} = \theta$ and $\mathrm{B\hat{C}A} = \frac{\pi}{3}$.

a. Show that length $\mathrm{AB} = \frac{3}{\sqrt{3} \cos \theta + \sin \theta}$. [4]

b. Given that AB has a minimum value, determine the value of θ for which this occurs. [4]

Markscheme

a. any attempt to use sine rule ***M1***

$\frac{\mathrm{AB}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3}-\theta\right)}$ ***A1***

$= \frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta-\cos\frac{2\pi}{3}\sin\theta}$ ***A1***

Note: Condone use of degrees.

$= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta}$ ***A1***

$\frac{\mathrm{AB}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta}$

$\therefore \mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta+\sin\theta}$ ***AG***

[4 marks]

b. **METHOD 1**

$(\mathrm{AB})' = \frac{-3\left(-\sqrt{3}\sin\theta+\cos\theta\right)}{\left(\sqrt{3}\cos\theta+\sin\theta\right)^2}$ ***M1A1***

setting $(\mathrm{AB})' = 0$ ***M1***

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 2

$\mathrm{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3}-\theta\right)}$

AB minimum when $\sin\left(\frac{2\pi}{3}-\theta\right)$ is maximum ***M1***

$\sin\left(\frac{2\pi}{3}-\theta\right) = 1$ ***(A1)***

$\frac{2\pi}{3}-\theta = \frac{\pi}{2}$ ***M1***

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 3

shortest distance from B to AC is perpendicular to AC ***R1***

$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ***M1A2***

[4 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the functions *f* and *g* defined by $f(x) = 2^{\frac{1}{x}}$ and $g(x) = 4 - 2^{\frac{1}{x}}$, $x \neq 0$.

(a) Find the coordinates of *P*, the point of intersection of the graphs of *f* and *g* .

(b) Find the equation of the tangent to the graph of f at the point P.

Markscheme

(a) $2^{\frac{1}{x}} = 4 - 2^{\frac{1}{x}}$

attempt to solve the equation ***M1***

$x = 1$ ***A1***

so P is $(1, 2)$, as $f(1) = 2$ ***A1*** ***N1***

(b) $f'(x) = -\frac{1}{x^2} 2^{\frac{1}{x}} \ln 2$ ***A1***

attempt to substitute x-value found in part (a) into their $f'(x)$ ***M1***

$f'(1) = -2 \ln 2$

$y - 2 = -2 \ln 2(x - 1)$ (or equivalent) ***M1A1*** ***N0***

[7 marks]

Examiners report

Most candidates answered part (a) correctly although some candidates showed difficulty solving the equation using valid methods. Part (b) was less successful with many candidates failing to apply chain rule to obtain the derivative of the exponential function.

The curve C has equation $2x^2 + y^2 = 18$. Determine the coordinates of the four points on C at which the normal passes through the point $(1, 0)$.

Markscheme

$4x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{y}$ ***M1A1***

Note: Allow follow through on incorrect $\frac{\mathrm{d}y}{\mathrm{d}x}$ from this point.

gradient of normal at (a, b) is $\frac{b}{2a}$

Note: No further A marks are available if a general point is not used

equation of normal at (a, b) is $y - b = \frac{b}{2a}(x - a)$ $\left(\Rightarrow y = \frac{b}{2a}x + \frac{b}{2}\right)$ ***M1A1***

substituting $(1, 0)$ ***M1***

$b = 0$ or $a = -1$ ***A1A1***

four points are $(3,\ 0),\ (-3,\ 0),\ (-1,\ 4),\ (-1,\ -4)$ ***A1A1***

Note: Award ***A1A0*** for any two points correct.

[9 marks]

Examiners report

Many students were able to obtain the first marks in this question by implicit differentiation but few were able to complete the question successfully. There were a number of students obtaining the correct final answers, but could not be given the marks due to incorrect working. Most common was students giving the equation of the normal as $y - 0 = \frac{y}{2x}(x-1)$, instead of taking a general point e.g. (a, b)

Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$.

Markscheme

$2 + x - x^2 = 2 - 3x + x^2$ ***M1***

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x-2) = 0$

$\Rightarrow x = 0,\ x = 2$ ***A1A1***

Note: Accept graphical solution.

Award ***M1*** for correct graph and ***A1A1*** for correctly labelled roots.

$\therefore \text{A} = \int_0^2 \left((2 + x - x^2) - (2 - 3x + x^2)\right) \mathrm{d}x$ ***(M1)***

$= \int_0^2 (4x - 2x^2)\mathrm{d}x$ or equivalent ***A1***

$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$ ***A1***

$= \frac{8}{3}\left(= 2\frac{2}{3}\right)$ ***A1***

[7 marks]

Examiners report

This was the question that gained the most correct responses. A few candidates struggled to find the limits of the integration or found a negative area.

Show that $\int_1^2 x^3 \ln x \mathrm{d}x = 4 \ln 2 - \frac{15}{16}$.

Markscheme

any attempt at integration by parts ***M1***

$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***(A1)***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^3 \Rightarrow v = \frac{x^4}{4}$ ***(A1)***

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{4}\mathrm{d}x$ ***A1***

Note: Condone absence of limits at this stage.

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \left[\frac{x^4}{16}\right]_1^2$ ***A1***

Note: Condone absence of limits at this stage.

$= 4\ln 2 - \left(1 - \frac{1}{16}\right)$ ***A1***

$= 4\ln 2 - \frac{15}{16}$ ***AG***

[6 marks]

Examiners report

[N/A]

Find the exact value of $\int_1^2 \left((x-2)^2 + \frac{1}{x} + \sin \pi x\right)$ dx.

Markscheme

$\left[\frac{1}{3}(x-2)^3 + \ln x - \frac{1}{\pi}\cos \pi x\right]_{(1)}^{(2)}$ ***A1A1A1***

Note: Accept $\frac{1}{3}x^3 - 2x^2 + 4x$ in place of $\frac{1}{3}(x-2)^3$.

$= \left(0 + \ln 2 - \frac{1}{\pi}\cos 2\pi\right) - \left(-\frac{1}{3} + \ln 1 - \frac{1}{\pi}\cos \pi\right)$ ***(M1)***

$= \frac{1}{3} + \ln 2 - \frac{2}{\pi}$ ***A1A1***

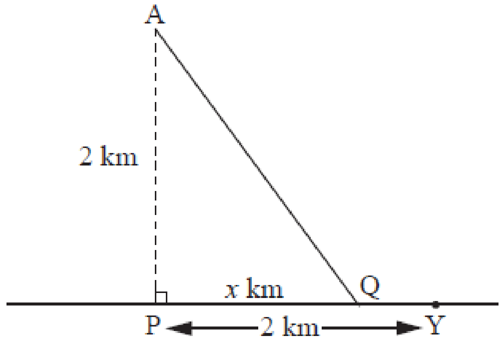
Note: Award ***A1*** for any two terms correct, ***A1*** for the third correct.

[6 marks]

Examiners report

Generally well done, although quite a number of candidates were either unable to integrate the sine term or incorrectly evaluated the resulting cosine at the limits.

André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \leqslant x \leqslant 2$, find an expression for the time T minutes taken by André to reach point Y.

(b) Show that $\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$.

(c) (i) Solve $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$.

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = \frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

Markscheme

(a) $\mathrm{AQ} = \sqrt{x^2+4}\ (\mathrm{km})$ ***(A1)***

$\mathrm{QY} = (2-x)\ (\mathrm{km})$ ***(A1)***

$T = 5\sqrt{5}\mathrm{AQ} + 5\mathrm{QY}$ ***(M1)***

$= 5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)\ (\mathrm{mins})$ ***A1***

[4 marks]

(b) Attempting to use the chain rule on $5\sqrt{5}\sqrt{(x^2+4)}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(5\sqrt{5}\sqrt{(x^2+4)}\right) = 5\sqrt{5} \times \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \times 2x$ ***A1***

$\left(= \frac{5\sqrt{5}x}{\sqrt{x^2+4}}\right)$

$\frac{\mathrm{d}}{\mathrm{d}x}(5(2-x)) = -5$ ***A1***

$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ ***AG N0***

[3 marks]

(c) (i) $\sqrt{5}x = \sqrt{x^2+4}$ ***A1***

Squaring both sides and rearranging to obtain $5x^2 = x^2 + 4$ ***M1***

$x = 1$ ***A1 N1***

Note: Do not award the final ***A1*** for stating a negative solution in final answer.

(ii) $T = 5\sqrt{5}\sqrt{1+4} + 5(2-1)$ ***M1***

= 30 (mins) ***A1 N1***

Note: Allow ***FT*** on incorrect x value.

(iii) **METHOD 1**

Attempting to use the quotient rule ***M1***

$u = x$, $v = \sqrt{x^2+4}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x(x^2+4)^{-1/2}$ ***(A1)***

$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 5\sqrt{5}\left[\frac{\sqrt{x^2+4}-\frac{1}{2}(x^2+4)^{-1/2}\times 2x^2}{(x^2+4)}\right]$ ***A1***

Attempt to simplify ***(M1)***

$= \frac{5\sqrt{5}}{(x^2+4)^{3/2}}\left[x^2+4-x^2\right]$ or equivalent ***A1***

$= \frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ ***AG***

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum ***R1*** ***N0***

Note: Allow ***FT*** on incorrect x value, $0 \leqslant x \leqslant 2$.

METHOD 2

Attempting to use the product rule ***M1***

$u = x$, $v = \sqrt{x^2+4}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x(x^2+4)^{-1/2}$ ***(A1)***

$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 5\sqrt{5}(x^2+4)^{-1/2} - \frac{5\sqrt{5}x}{2}(x^2+4)^{-3/2} \times 2x$ ***A1***

$\left(= \frac{5\sqrt{5}}{(x^2+4)^{1/2}} - \frac{5\sqrt{5}x^2}{(x^2+4)^{3/2}}\right)$

Attempt to simplify ***(M1)***

$= \frac{5\sqrt{5}(x^2+4)-5\sqrt{5}x^2}{(x^2+4)^{3/2}}$ $\left(= \frac{5\sqrt{5}(x^2+4-x^2)}{(x^2+4)^{3/2}}\right)$ ***A1***

$= \frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ ***AG***

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum ***R1*** ***N0***

Note: Allow ***FT*** on incorrect x value, $0 \leqslant x \leqslant 2$.

[11 marks]

Total [18 marks]

Examiners report

Most candidates scored well on this question. The question tested their competence at algebraic manipulation and differentiation. A few candidates failed to extract from the context the correct relationship between velocity, distance and time.

(a) Given that $\alpha > 1$, use the substitution $u = \frac{1}{x}$ to show that

$$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}x.$$

(b) **Hence** show that $\arctan\alpha + \arctan\frac{1}{\alpha} = \frac{\pi}{2}$.

Markscheme

(a) $u = \frac{1}{x} \Rightarrow \mathrm{d}u = -\frac{1}{x^2}\mathrm{d}x$ ***M1***

$\Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{u^2}$ ***A1***

$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = -\int_1^{\frac{1}{\alpha}} \frac{1}{1+\left(\frac{1}{u}\right)^2}\frac{\mathrm{d}u}{u^2}$ ***A1M1A1***

Note: Award ***A1*** for correct integrand and ***M1A1*** for correct limits.

$= \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}u$ (upon interchanging the two limits) ***AG***

(b) $\arctan x^{\alpha}_{1} = \arctan u^{1}_{\frac{1}{\alpha}}$ ***A1***

$\arctan \alpha - \frac{\pi}{4} = \frac{\pi}{4} - \arctan \frac{1}{\alpha}$ ***A1***

$\arctan \alpha + \arctan \frac{1}{\alpha} = \frac{\pi}{2}$ ***AG***

[7 marks]

Examiners report

This question was successfully answered by few candidates. Both parts of the question prescribed the approach which was required – "use the substitution" and "hence". Many candidates ignored these. The majority of the candidates failed to use substitution properly to change the integration variables and in many cases the limits were fudged. The logic of part (b) was missing in many cases.

The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

a. Find the value of p and the value of q . [4]

b. The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. [2]

Markscheme

a. **METHOD 1**

$f'(x) = q - 2x = 0$ ***M1***

$f'(3) = q - 6 = 0$

$q = 6$ ***A1***

$f(3) = p + 18 - 9 = 5$ ***M1***

$p = -4$ ***A1***

METHOD 2

$f(x) = -(x-3)^2 + 5$ ***M1A1***

$= -x^2 + 6x - 4$

$q = 6, p = -4$ ***A1A1***

[4 marks]

b. $g(x) = -4 + 6(x-3) - (x-3)^2 \; (= -31 + 12x - x^2)$ ***M1A1***

Note: Accept any alternative form which is correct.

Award ***M1A0*** for a substitution of $(x + 3)$.

[2 marks]

Examiners report

a. In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

b. In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

---

a. A particle P moves in a straight line with displacement relative to origin given by [10]

$$s = 2\sin(\pi t) + \sin(2\pi t),\ t \geqslant 0,$$

where t is the time in seconds and the displacement is measured in centimetres.

(i) Write down the period of the function s.

(ii) Find expressions for the velocity, v, and the acceleration, a, of P.

(iii) Determine all the solutions of the equation $v = 0$ for $0 \leqslant t \leqslant 4$.

b. Consider the function [8]

$$f(x) = A\sin(ax) + B\sin(bx),\ A,\ a,\ B,\ b,\ x \in \mathbb{R}.$$

Use mathematical induction to prove that the $(2n)^{\text{th}}$ derivative of f is given by $\left(f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)\right.$, for all $n \in \mathbb{Z}^+$.

Markscheme

a. (i) the period is 2 ***A1***

(ii) $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 2\pi\cos(\pi t) + 2\pi\cos(2\pi t)$ ***(M1)A1***

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -2\pi^2\sin(\pi t) - 4\pi^2\sin(2\pi t)$ ***(M1)A1***

(iii) $v = 0$

$2\pi\left(\cos(\pi t) + \cos(2\pi t)\right) = 0$

EITHER

$\cos(\pi t) + 2\cos^2(\pi t) - 1 = 0$ ***M1***

$\left(2\cos(\pi t) - 1\right)\left(\cos(\pi t) + 1\right) = 0$ ***(A1)***

$\cos(\pi t) = \frac{1}{2}$ or $\cos(\pi t) = -1$ ***A1***

$t = \frac{1}{3},\ t = 1$ ***A1***

$t = \frac{5}{3},\ t = \frac{7}{3},\ t = \frac{11}{3},\ t = 3$ ***A1***

OR

$2\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right) = 0$ ***M1***

$\cos\left(\frac{\pi t}{2}\right) = 0$ or $\cos\left(\frac{3\pi t}{2}\right) = 0$ ***A1A1***

$t = \frac{1}{3},\ 1$ ***A1***

$t = \frac{5}{3},\ \frac{7}{3},\ 3,\ \frac{11}{3}$ ***A1***

[10 marks]

b. $P(n) : f^{(2n)}(x) = (-1)^n\left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)$

$P(1) : f''(x) = (Aa\cos(ax) + Bb\cos(bx))'$ ***M1***

$= -Aa^2\sin(ax) - Bb^2\sin(bx)$

$= -1\left(Aa^2\sin(ax) + Bb^2\sin(bx)\right)$ ***A1***

$\therefore P(1)$ true

assume that

$P(k) : f^{(2k)}(x) = (-1)^k\left(Aa^{2k}\sin(ax) + Bb^{2k}\sin(bx)\right)$ is true ***M1***

consider $P(k+1)$

$f^{(2k+1)}(x) = (-1)^k\left(Aa^{2k+1}\cos(ax) + Bb^{2k+1}\cos(bx)\right)$ ***M1A1***

$f^{(2k+2)}(x) = (-1)^k\left(-Aa^{2k+2}\sin(ax) - Bb^{2k+2}\sin(bx)\right)$ ***A1***

$= (-1)^{k+1}\left(Aa^{2k+2}\sin(ax) + Bb^{2k+2}\sin(bx)\right)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true $\forall n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** only if the previous three ***M*** marks have been awarded.

[8 marks]

Examiners report

a. In (a), only a few candidates gave the correct period but the expressions for velocity and acceleration were correctly obtained by most candidates. In (a)(iii), many candidates manipulated the equation $v = 0$ correctly to give the two possible values for $\cos(\pi t)$ but then failed to find all the possible values of t.

b. Solutions to (b) were disappointing in general with few candidates giving a correct solution.

The graph below shows the two curves $y = \frac{1}{x}$ and $y = \frac{k}{x}$, where $k > 1$.

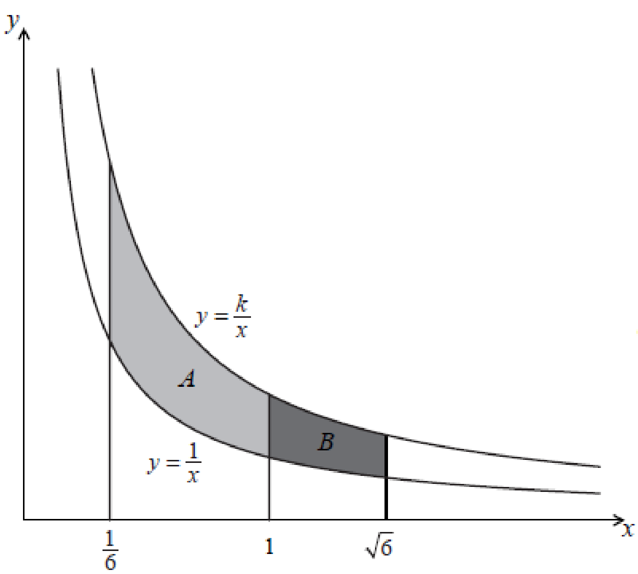


diagram not to scale

a. Find the area of region A in terms of k . [3]

b. Find the area of region B in terms of k . [2]

c. Find the ratio of the area of region A to the area of region B . [3]

Markscheme

a. $\int_{\frac{1}{6}}^{1} \frac{k}{x} - \frac{1}{x} \mathrm{d}x = (k-1)[\ln x]_{\frac{1}{6}}^{1}$ ***M1*** ***A1***

Note: Award ***M1*** for $\int \frac{k}{x} - \frac{1}{x} \mathrm{d}x$ or $\int \frac{1}{x} - \frac{k}{x} \mathrm{d}x$ and ***A1*** for $(k-1)\ln x$ seen in part (a) or later in part (b).

$= (1-k)\ln \frac{1}{6}$ ***A1***

[3 marks]

b. $\int_{1}^{\sqrt{6}} \frac{k}{x} - \frac{1}{x} \mathrm{d}x = (k-1)[\ln x]_{1}^{\sqrt{6}}$ ***(A1)***

Note: Award ***A1*** for correct change of limits.

$= (k-1)\ln \sqrt{6}$ ***A1***

[2 marks]

c. $(1-k)\ln \frac{1}{6} = (k-1)\ln 6$ ***A1***

$(k-1)\ln \sqrt{6} = \frac{1}{2}(k-1)\ln 6$ ***A1***

Note: This simplification could have occurred earlier, and marks should still be awarded.

ratio is 2 (or 2:1) ***A1***

[3 marks]

Examiners report

a. Generally well answered by most candidates. Basic algebra sometimes let students down in the simplification of the ratio in part (c). It was not uncommon to see $\frac{\log A}{\log B}$ simplified to $\frac{A}{B}$.

b. Generally well answered by most candidates. Basic algebra sometimes let students down in the simplification of the ratio in part (c). It was not uncommon to see $\frac{\log A}{\log B}$ simplified to $\frac{A}{B}$.

c. Generally well answered by most candidates. Basic algebra sometimes let students down in the simplification of the ratio in part (c). It was not uncommon to see $\frac{\log A}{\log B}$ simplified to $\frac{A}{B}$.

Let $f(x) = \frac{2-3x^5}{2x^3},\ x \in \mathbb{R},\ x \neq 0.$

a. The graph of $y = f(x)$ has a local maximum at A. Find the coordinates of A. [5]

b.i. Show that there is exactly one point of inflexion, B, on the graph of $y = f(x)$. [5]

b.ii. The coordinates of B can be expressed in the form B$\left(2^a,\ b \times 2^{-3a}\right)$ where *a*, *b*$\in \mathbb{Q}$. Find the value of *a* and the value of *b*. [3]

c. Sketch the graph of $y = f(x)$ showing clearly the position of the points A and B. [4]

Markscheme

a. attempt to differentiate ***(M1)***

$f'(x) = -3x^{-4} - 3x$ ***A1***

Note: Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example $f'(x) = \frac{-15x^4 \times 2x^3 - 6x^2(2-3x^5)}{(2x^3)^2}$.

$-\frac{3}{x^4} - 3x = 0$ ***M1***

$\Rightarrow x^5 = -1 \Rightarrow x = -1$ ***A1***

$\mathrm{A}\left(-1, -\frac{5}{2}\right)$ ***A1***

[5 marks]

b.i. $f''(x) = 0$ ***M1***

$f''(x) = 12x^{-5} - 3\,(= 0)$ ***A1***

Note: Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***A1***

hence (at most) one point of inflexion ***R1***

Note: This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''(x)$ changes sign at $x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***R1***

so exactly one point of inflexion

[5 marks]

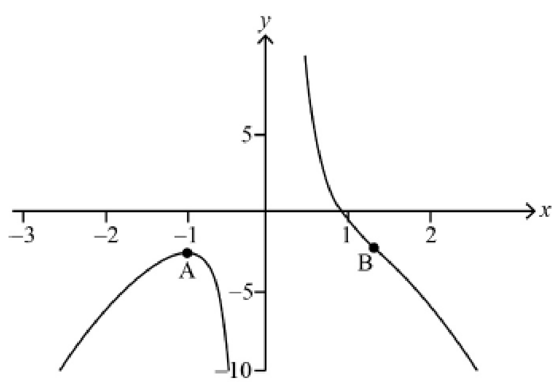
b.ii $x = \sqrt[5]{4} = 2^{\frac{2}{5}}\left(\Rightarrow a = \frac{2}{5}\right)$ ***A1***

$f\left(2^{\frac{2}{5}}\right) = \frac{2-3\times 2^2}{2\times 2^{\frac{6}{5}}} = -5 \times 2^{-\frac{6}{5}}\,(\Rightarrow b = -5)$ ***(M1)A1***

Note: Award ***M1*** for the substitution of their value for x into $f(x)$.

[3 marks]

c.



A1A1A1A1

A1 for shape for *x* < 0
A1 for shape for *x* > 0
A1 for maximum at A
A1 for POI at B.

Note: Only award last two ***A1***s if A and B are placed in the correct quadrants, allowing for follow through.

[4 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
[N/A]

c.

A normal to the graph of $y = \arctan(x-1)$, for $x > 0$, has equation $y = -2x + c$, where $x \in \mathbb{R}$.

Find the value of c.

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x-1)) = \frac{1}{1+(x-1)^2}$ (or equivalent) ***A1***

$m_N = -2$ and so $m_T = \frac{1}{2}$ ***(R1)***

Attempting to solve $\frac{1}{1+(x-1)^2} = \frac{1}{2}$ (or equivalent) for x ***M1***

$x = 2$ (as $x > 0$) ***A1***

Substituting $x = 2$ and $y = \frac{\pi}{4}$ to find c ***M1***

$c = 4 + \frac{\pi}{4}$ ***A1 N1***

[6 marks]

Examiners report

There was a disappointing response to this question from a fair number of candidates. The differentiation was generally correctly performed, but it was then often equated to $-2x + c$ rather than the correct numerical value. A few candidates either didn't simplify arctan(1) to $\frac{\pi}{4}$, or stated it to be 45 or $\frac{\pi}{2}$.

The function *f* is defined by $f(x) = \frac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

a. Express $f(x)$ in the form $A + \frac{B}{x+2}$, where A and $B \in \mathbb{Z}$. [2]

b. Hence show that $f'(x) > 0$ on D. [2]

c. State the range of *f*. [2]

d. (i) Find an expression for $f^{-1}(x)$. [8]

(ii) Sketch the graph of $y = f(x)$, showing the points of intersection with both axes.

(iii) On the same diagram, sketch the graph of $y = f'(x)$.

e. (i) On a different diagram, sketch the graph of $y = f(|x|)$ where $x \in D$. [7]

(ii) Find all solutions of the equation $f(|x|) = -\frac{1}{4}$.

Markscheme

a. by division or otherwise

$f(x) = 2 - \frac{5}{x+2}$ ***A1A1***

[2 marks]

b. $f'(x) = \frac{5}{(x+2)^2}$ ***A1***

> 0 as $(x+2)^2 > 0$ (on D) ***R1AG***

Note: Do not penalise candidates who use the original form of the function to compute its derivative.

[2 marks]

c. $S = \left[-3, \frac{3}{2}\right]$ ***A2***

Note: Award ***A1A0*** for the correct endpoints and an open interval.

[2 marks]

d. (i) **EITHER**

rearrange $y = f(x)$ to make *x* the subject ***M1***

obtain one-line equation, *e.g.* $2x - 1 = xy + 2y$ ***A1***

$x = \frac{2y+1}{2-y}$ ***A1***

OR

interchange *x* and *y* ***M1***

obtain one-line equation, *e.g.* $2y - 1 = xy + 2x$ ***A1***

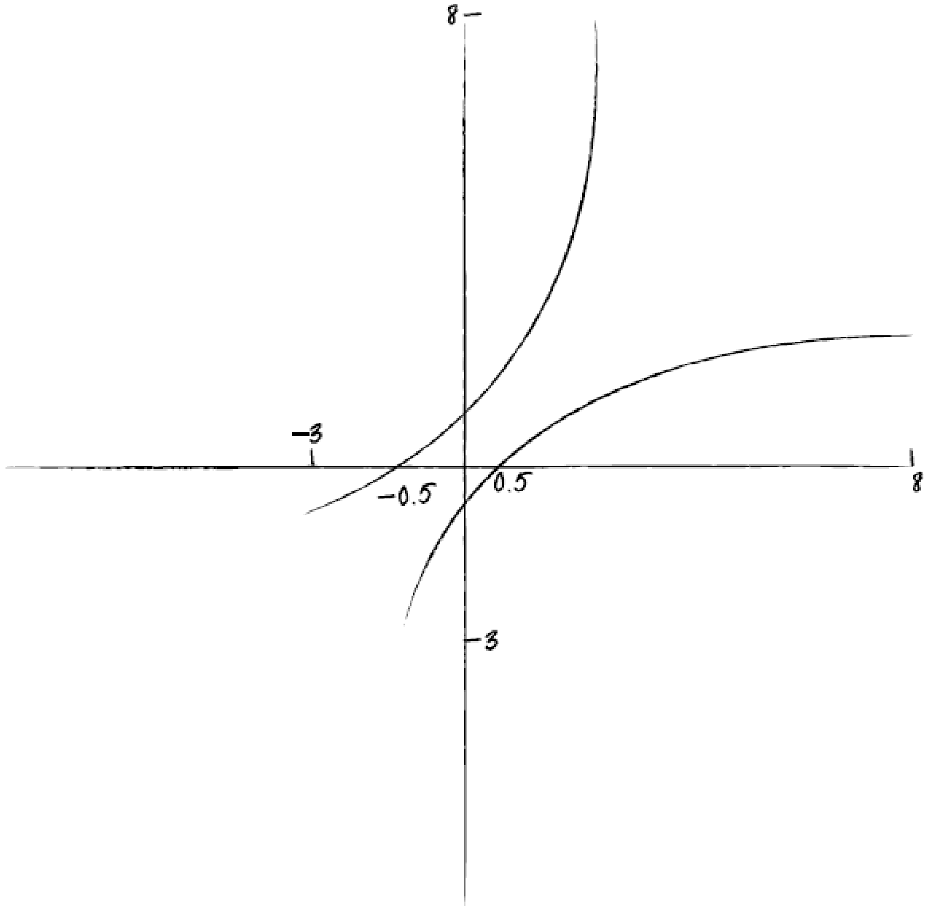
$y = \frac{2x+1}{2-x}$ ***A1***

THEN

$f^{-1}(x) = \frac{2x+1}{2-x}$ ***A1***

Note: Accept $\frac{5}{2-x} - 2$

(ii), (iii)



A1A1A1A1

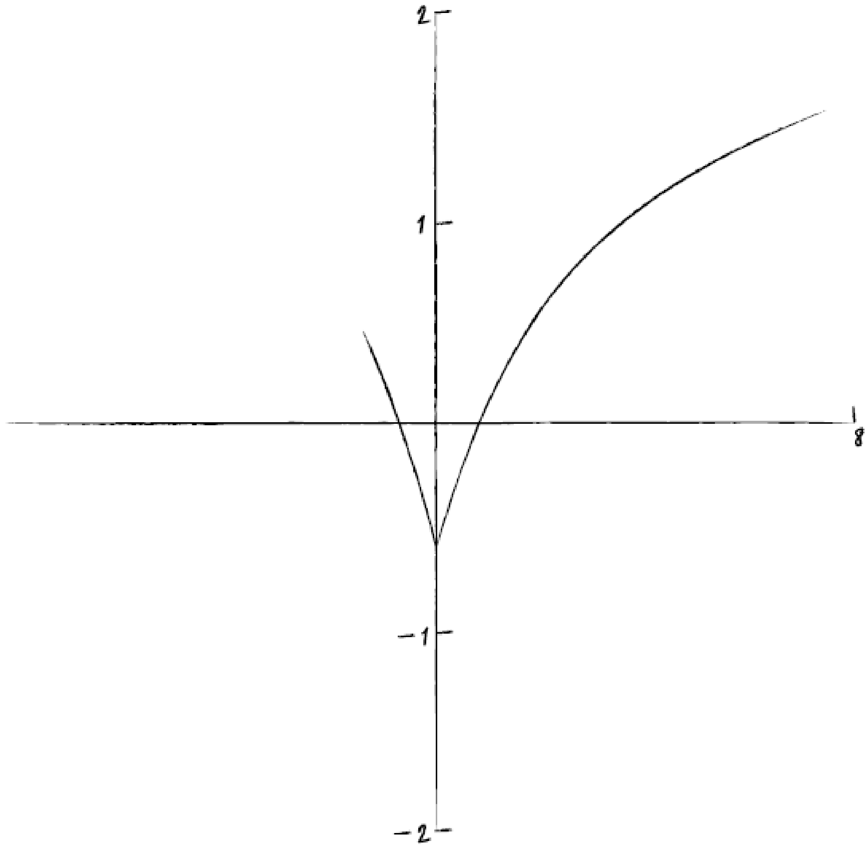
[8 marks]

Note: Award ***A1*** for correct shape of $y = f(x)$.

Award ***A1*** for x intercept $\frac{1}{2}$ seen. Award ***A1*** for y intercept $-\frac{1}{2}$ seen.

Award ***A1*** for the graph of $y = f^{-1}(x)$ being the reflection of $y = f(x)$ in the line $y = x$. Candidates are not required to indicate the full domain, but $y = f(x)$ should not be shown approaching $x = -2$. Candidates, in answering (iii), can **FT** on their sketch in (ii).

e. (i)



A1A1A1

Note: ***A1*** for correct sketch $x > 0$, ***A1*** for symmetry, ***A1*** for correct domain (from –1 to +8).

Note: Candidates can **FT** on their sketch in (d)(ii).

(ii) attempt to solve $f(x) = -\frac{1}{4}$ ***(M1)***

obtain $x = \frac{2}{9}$ ***A1***

use of symmetry or valid algebraic approach ***(M1)***

obtain $x = -\frac{2}{9}$ ***A1***

[7 marks]

Examiners report

a. Generally well done.

b. In their answers to Part (b), most candidates found the derivative, but many assumed it was obviously positive.

c. [N/A]

d. Part (d)(i) Generally well done, but some candidates failed to label their final expression as $f^{-1}(x)$. Part (d)(ii) Marks were lost by candidates who failed to mark the intercepts with values.

e. Marks were also lost in this part and in part (e)(i) for graphs that went beyond the explicitly stated domain.

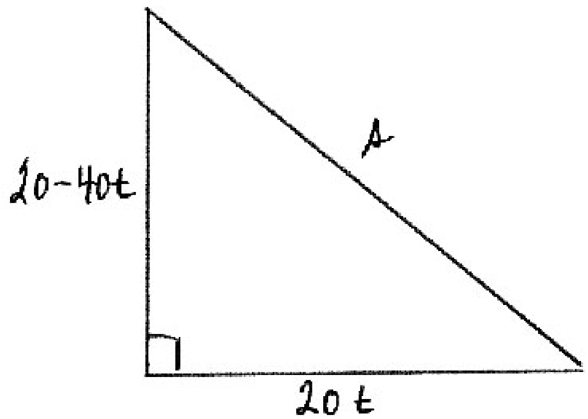
At 12:00 a boat is 20 km due south of a freighter. The boat is travelling due east at 20 km h^{-1}, and the freighter is travelling due south at 40 km h^{-1}.

a. Determine the time at which the two ships are closest to one another, and justify your answer. [8]

b. If the visibility at sea is 9 km, determine whether or not the captains of the two ships can ever see each other's ship. [3]

Markscheme

a.



(M1)

$s^2 = (20t)^2 + (20 - 40t)^2$ ***M1***

$s^2 = 2000t^2 - 1600t + 400$ ***A1***

to minimize s it is enough to minimize s^2

$f'(t) = 4000t - 1600$ ***A1***

setting $f'(t)$ equal to 0 ***M1***

$4000t - 1600 = 0 \Rightarrow t = \frac{2}{5}$ or 24 minutes ***A1***

$f''(t) = 4000 > 0$ ***M1***

$\Rightarrow$ at $t = \frac{2}{5}$, $f(t)$ is minimized

hence, the ships are closest at 12:24 ***A1***

Note: accept solution based on s.

[8 marks]

b. $f\left(\frac{2}{5}\right) = \sqrt{80}$ ***M1A1***

since $\sqrt{80} < 9$, the captains can see one another ***R1***

[3 marks]

Examiners report

a. This was, disappointingly, a poorly answered question. Some tried to talk their way through the question without introducing the time variable. Even those who did use the distance as a function of time often did not check for a minimum.

b. This was, disappointingly, a poorly answered question. Some tried to talk their way through the question without introducing the time variable. Even those who did use the distance as a function of time often did not check for a minimum.

The function *f* is defined by $f(x)=\frac{1}{4x^2-4x+5}$.

a. Express $4x^2-4x+5$ in the form $a(x-h)^2+k$ where $a, h, k\in\mathbb{Q}$. [2]

b. The graph of $y=x^2$ is transformed onto the graph of $y=4x^2-4x+5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3]

c. Sketch the graph of $y=f(x)$. [2]

d. Find the range of *f*. [2]

e. By using a suitable substitution show that $\int f(x)\mathrm{d}x=\frac{1}{4}\int\frac{1}{u^2+1}\mathrm{d}u$. [3]

f. Prove that $\int_1^{3.5}\frac{1}{4x^2-4x+5}\mathrm{d}x=\frac{\pi}{16}$. [7]

Markscheme

a. $4(x-0.5)^2+4$ ***A1A1***

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

[2 marks]

b. translation $\begin{pmatrix}0.5\\0\end{pmatrix}$ (allow "0.5 to the right") ***A1***

stretch parallel to *y*-axis, scale factor 4 (allow vertical stretch or similar) ***A1***

translation $\begin{pmatrix}0\\4\end{pmatrix}$ (allow "4 up") ***A1***

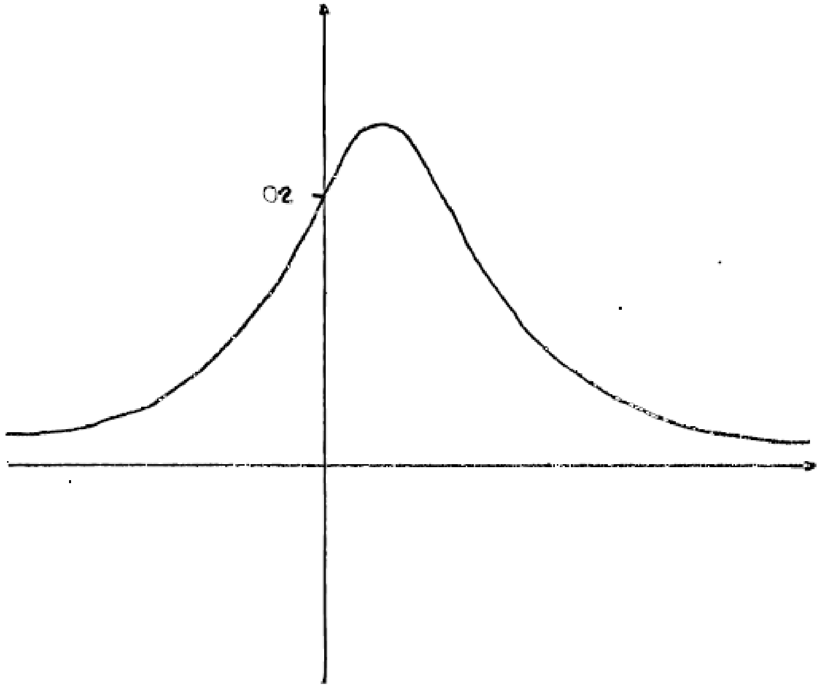
Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation of $\begin{pmatrix}0.5\\4\end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix}0\\1\end{pmatrix}$.

[3 marks]

c.



general shape (including asymptote and single maximum in first quadrant), ***A1***

intercept $\left(0, \frac{1}{5}\right)$ or maximum $\left(\frac{1}{2}, \frac{1}{4}\right)$ shown ***A1***

[2 marks]

d. $0 < f(x) \leqslant \frac{1}{4}$ ***A1A1***

Note: ***A1*** for $\leqslant \frac{1}{4}$, ***A1*** for $0 <$.

[2 marks]

e. let $u = x - \frac{1}{2}$ ***A1***

$\frac{du}{dx} = 1 \quad (\text{or } du = dx)$ ***A1***

$\int \frac{1}{4x^2-4x+5} dx = \int \frac{1}{4\left(x-\frac{1}{2}\right)^2+4} dx$ ***A1***

$\int \frac{1}{4u^2+4} du = \frac{1}{4} \int \frac{1}{u^2+1} du$ ***AG***

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

f. $\int_1^{3.5} \frac{1}{4x^2-4x+5} dx = \frac{1}{4} \int_{0.5}^{3} \frac{1}{u^2+1} du$ ***A1***

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to *x* values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}[\arctan(u)]_{0.5}^{3}$ ***(M1)***

$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***A1***

let the integral = I

$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***M1***

$\frac{3-0.5}{1+3\times 0.5} = \frac{2.5}{2.5} = 1$ ***(M1)A1***

$4I = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{16}$ ***A1AG***

[7 marks]

Examiners report

a. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

b. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

   (b) Exam technique would have helped those candidates who could not get part (a) correct as any solution of the form given in the question could have led to full marks in part (b). Several candidates obtained expressions which were not of this form in (a) and so were unable to receive any marks in (b) Many missed the fact that if a vertical translation is performed before the vertical stretch it has a different magnitude to if it is done afterwards. Though on this occasion the markscheme was fairly flexible in the words it allowed to be used by candidates to describe the transformations it would be less risky to use the correct expressions.

c. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

   (c) Generally the sketches were poor. The general rule for all sketch questions should be that any asymptotes or intercepts should be clearly labelled. Sketches do not need to be done on graph paper, but a ruler should be used, particularly when asymptotes are involved.

d. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

e. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

   (e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

f. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

   (e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

A curve has equation $x^3y^2 + x^3 - y^3 + 9y = 0$. Find the coordinates of the three points on the curve where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Markscheme

$3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 9\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1M1A1***

Note: First ***M1*** for attempt at implicit differentiation, second ***M1*** for use of product rule.

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2y^2+3x^2}{3y^2-2x^3y-9}\right)$
$\Rightarrow 3x^2 + 3x^2y^2 = 0$ ***(A1)***
$\Rightarrow 3x^2\left(1+y^2\right) = 0$
$x = 0$ ***A1***

Note: Do not award ***A1*** if extra solutions given *eg* $y = \pm 1$.

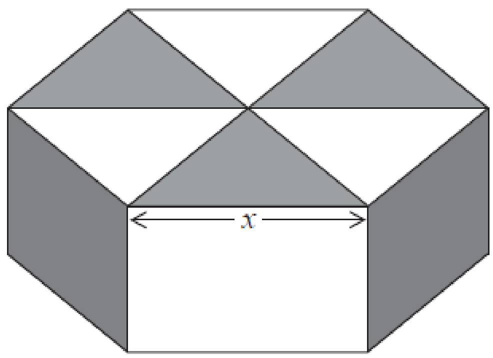
substituting $x = 0$ into original equation ***(M1)***

$y^3 - 9y = 0$
$y(y+3)(y-3) = 0$
$y = 0,\ y = \pm 3$
coordinates $(0,0), (0,3), (0,-3)$ ***A1***
[7 marks]

Examiners report

The majority of candidates were able to apply implicit differentiation and the product rule correctly to obtain $3x^2\left(1+y^2\right) = 0$. The better then recognised that $x = 0$ was the only possible solution. Such candidates usually went on to obtain full marks. A number decided that $y = \pm 1$ though then made no further progress. The solution set $x = 0$ and $y = \pm i$ was also occasionally seen. A small minority found the correct x and y values for the three co-ordinates but then surprisingly expressed them as $(0, 0)$, $(3, 0)$ and $(-3, 0)$.

A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.



(a) Show that the area of each hexagon is $\frac{3\sqrt{3}x^2}{2}\text{cm}^2$.
(b) Given that the volume of the box is 90 cm^2 , show that when $x = \sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum.

Markscheme

(a) Area of hexagon $= 6 \times \frac{1}{2} \times x \times x \times \sin 60^\circ$ ***M1***
$= \frac{3\sqrt{3}x^2}{2}$ ***AG***

(b) Let the height of the box be h
Volume $= \frac{3\sqrt{3}hx^2}{2} = 90$ ***M1***
Hence $h = \frac{60}{\sqrt{3}x^2}$ ***A1***
Surface area, $A = 3\sqrt{3}x^2 + 6hx$ ***M1***
$= 3\sqrt{3}x^2 + \frac{360}{\sqrt{3}}x^{-1}$ ***A1***
$\frac{\mathrm{d}A}{\mathrm{d}x} = 6\sqrt{3}x - \frac{360}{\sqrt{3}}x^{-2}$ ***A1***
$\left(\frac{\mathrm{d}A}{\mathrm{d}x} = 0\right)$
$6\sqrt{3}x^3 = \frac{360}{\sqrt{3}}$ ***M1***
$x^3 = 20$

$x = \sqrt[3]{20}$ ***AG***

$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = 6\sqrt{3} + \frac{720x^{-3}}{\sqrt{3}}$

which is positive when $x = \sqrt[3]{20}$, and hence gives a minimum value. ***R1***

[8 marks]

Examiners report

There were a number of wholly correct answers seen and the best candidates tackled the question well. However, many candidates did not seem to understand what was expected in such a problem. It was disappointing that a significant number of candidates were unable to find the area of the hexagon.

The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

a. State the two zeros of f. [1]

b. Sketch the graph of f. [1]

c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is [7]

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}.$$

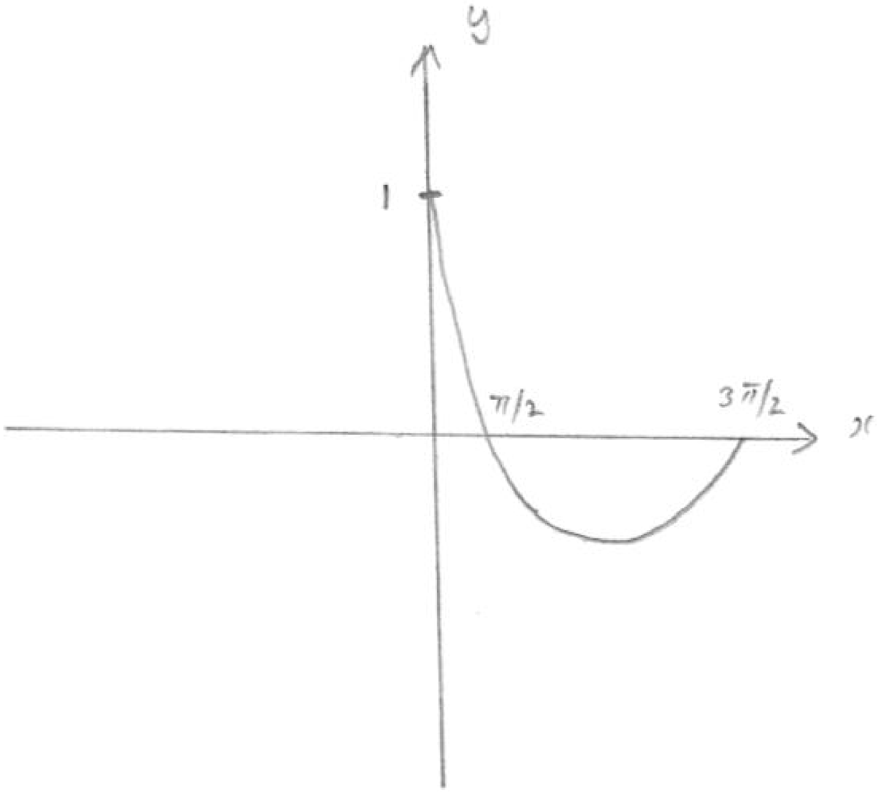
Markscheme

a. $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ ***A1***

[1 mark]

b.



A1

Note: Accept any form of concavity for $x \in \left[0, \frac{\pi}{2}\right]$.

Note: Do not penalize unmarked zeros if given in part (a).

Note: Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

c. attempt at integration by parts ***M1***

EITHER

$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = -\mathrm{e}^{-x} \cos x \mathrm{d}x - \int \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***A1***

$\Rightarrow I = -\mathrm{e}^{-x} \cos x \mathrm{d}x - \left[-\mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \cos x \mathrm{d}x\right]$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

OR

$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = \mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***A1***

$I = \mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - \int \mathrm{e}^{-x} \cos x \mathrm{d}x$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

THEN

$\int_0^{\frac{\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ ***A1***

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2} - \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}$ ***A1***

ratio of *A*:*B* is $\frac{\frac{\mathrm{e}^{-\frac{\pi}{2}}}{2} + \frac{1}{2}}{\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2} + \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}}$

$= \frac{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{\pi}{2}} + 1\right)}{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{3\pi}{2}} + \mathrm{e}^{-\frac{\pi}{2}}\right)}$ ***M1***

$= \frac{\mathrm{e}^{\pi}\left(\mathrm{e}^{\frac{\pi}{2}} + 1\right)}{\mathrm{e}^{\pi} + 1}$ ***AG***

[7 marks]

Examiners report

a. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

b. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

c. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

The function *f* is defined on the domain $x \geqslant 0$ by $f(x) = \mathrm{e}^x - x^{\mathrm{e}}$.

a. (i) Find an expression for $f'(x)$. [3]

(ii) Given that the equation $f'(x) = 0$ has two roots, state their values.

b. Sketch the graph of f , showing clearly the coordinates of the maximum and minimum. [3]

c. Hence show that $\mathrm{e}^\pi > \pi^{\mathrm{e}}$. [1]

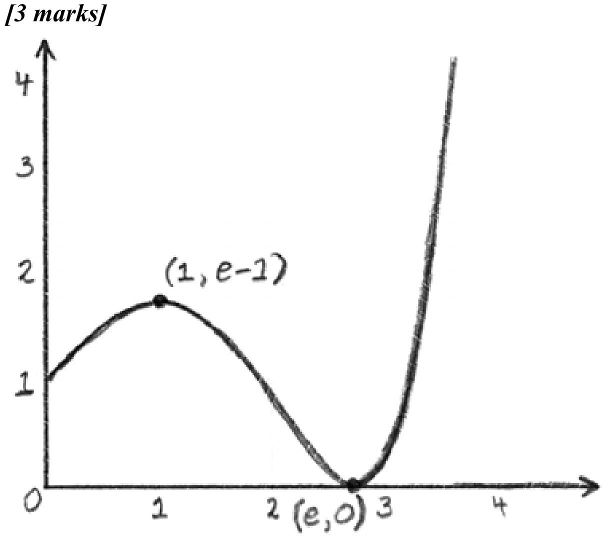
Markscheme

a. (i) $f'(x) = \mathrm{e}^x - \mathrm{e}x^{\mathrm{e}-1}$ ***A1***

(ii) by inspection the two roots are 1, e ***A1A1***

[3 marks]

b.



A3

Note: Award ***A1*** for maximum, ***A1*** for minimum and ***A1*** for general shape.

[3 marks]

c. from the graph: $e^x > x^e$ for all $x > 0$ except $x = e$ ***R1***

putting $x = \pi$, conclude that $e^\pi > \pi^e$ ***AG***

[1 mark]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function f is defined as $f(x) = e^{3x+1},\ x \in \mathbb{R}$.

a. (i) Find $f^{-1}(x)$. [4]

(ii) State the domain of f^{-1}.

b. The function g is defined as $g(x) = \ln x,\ x \in \mathbb{R}^+$. [5]

The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

Find the coordinates of P.

c. The graph of $y = g(x)$ intersects the x-axis at the point Q. [3]

Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$.

d. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = e$. [5]

Find the area of the region R.

e. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = e$. [6]

(i) Show that $g(x) \le x - 1,\ x \in \mathbb{R}^+$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$.

Markscheme

a. (i) $x = e^{3y+1}$ ***M1***

Note: The ***M1*** is for switching variables and can be awarded at any stage.

Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

b. $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = \mathrm{e}^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(\mathrm{e}^{-\frac{1}{2}},\ -\frac{1}{2}\right)$ ***A1***

[5 marks]

c. coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

at Q, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

d. let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$ ***A1***

[5 marks]

e. (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0 \quad \text{and} \quad h'(x) = 1 - \frac{1}{x}$ ***(A1)***

as $h'(x) \geq 0 \quad \text{for} \quad x \geq 1, \quad \text{then} \quad h(x) \geq 0 \quad \text{for} \quad x \geq 1$ ***R1***

as $h'(x) \leq 0 \quad \text{for} \quad 0 < x \leq 1, \quad \text{then} \quad h(x) \geq 0 \quad \text{for} \quad 0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\frac{1}{x^2}$ ***A1***

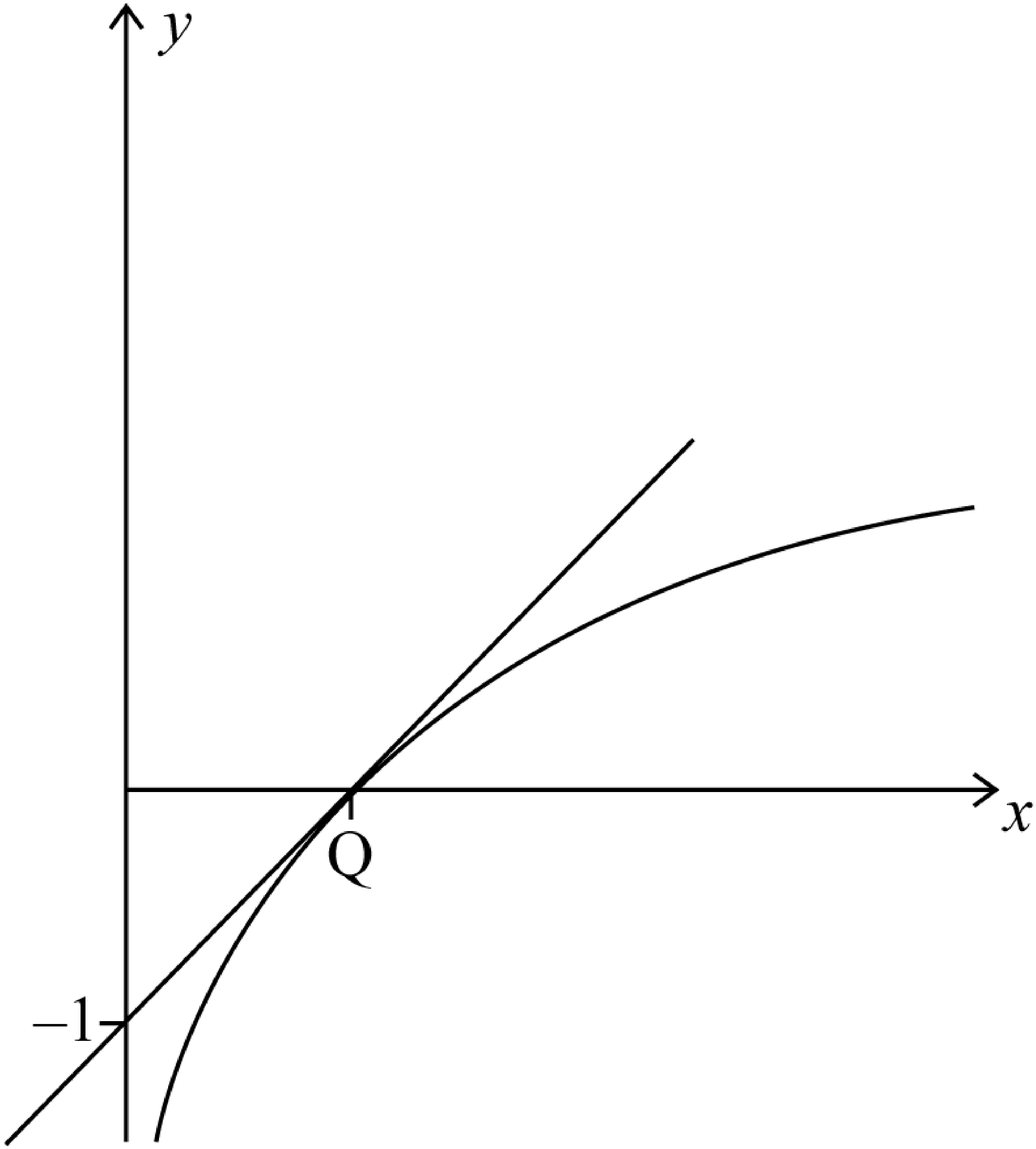
$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent $(y = x - 1 \quad \text{at} \quad x = 1)$ ***R1***

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

METHOD 3



clear correct graphs of $y = x - 1 \quad \text{and} \quad \ln x \quad \text{for} \quad x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \leq \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \leq \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \geq 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \leq g(x),\ x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

Examiners report

a. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

b. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

c. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

d. A productive question for many candidates, but some didn't realise that a difference of areas/integrals was required.

e. (i) Many candidates adopted a graphical approach, but sometimes with unconvincing reasoning.

(ii) Poorly answered. Many candidates applied the suggested substitution only to one side of the inequality, and then had to fudge the answer.

A function is defined as $f(x) = k\sqrt{x}$, with $k > 0$ and $x \geqslant 0$.

(a) Sketch the graph of $y = f(x)$.

(b) Show that *f* is a one-to-one function.

(c) Find the inverse function, $f^{-1}(x)$ and state its domain.

(d) If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point (4, 4) find the value of *k* .

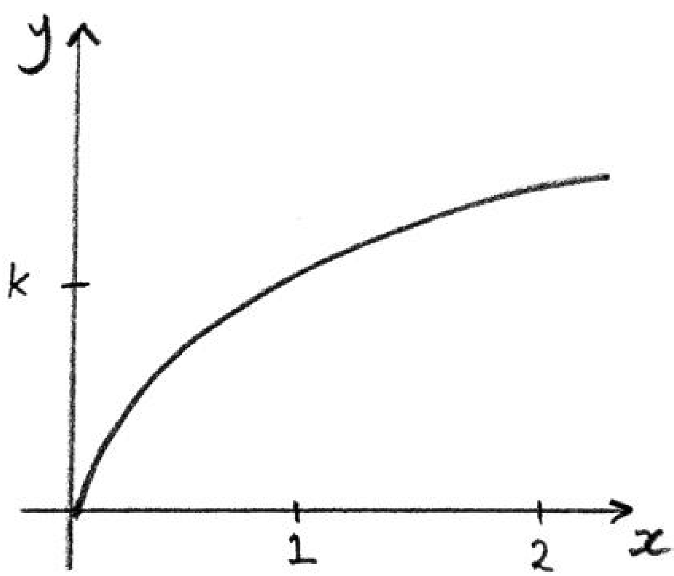
(e) Consider the graphs of $y = f(x)$ and $y = f^{-1}(x)$ using the value of *k* found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of $y = f(x)$ and $y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to $y = f(x)$ at point P is parallel to the tangent to $y = f^{-1}(x)$ at point Q find the value of *c* .

Markscheme

(a)



A1

Note: Award ***A1*** for correct concavity, passing through (0, 0) and increasing.

Scales need not be there.

[1 mark]

(b) a statement involving the application of the Horizontal Line Test or equivalent ***A1***

[1 mark]

(c) $y = k\sqrt{x}$

for either $x = k\sqrt{y}$ or $x = \frac{y^2}{k^2}$ ***A1***

$f^{-1}(x) = \frac{x^2}{k^2}$ ***A1***

$\text{dom}\left(f^{-1}(x)\right) = [0, \infty[$ ***A1***

[3 marks]

(d) $\frac{x^2}{k^2} = k\sqrt{x}$ or equivalent method ***M1***

$k = \sqrt{x}$

$k = 2$ ***A1***

[2 marks]

(e) (i) $A = \int_a^b (y_1 - y_2)\mathrm{d}x$ ***(M1)***

$A = \int_0^4 \left(2x^{\frac{1}{2}} - \frac{1}{4}x^2\right) \mathrm{d}x$ ***A1***

$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{12}x^3\right]_0^4$ ***A1***

$= \frac{16}{3}$ ***A1***

(ii) attempt to find either $f'(x)$ or $(f^{-1})'(x)$ ***M1***

$f'(x) = \frac{1}{\sqrt{x}}, \ \left((f^{-1})'(x) = \frac{x}{2}\right)$ ***A1A1***

$\frac{1}{\sqrt{c}} = \frac{c}{2}$ ***M1***

$c = 2^{\frac{2}{3}}$ ***A1***

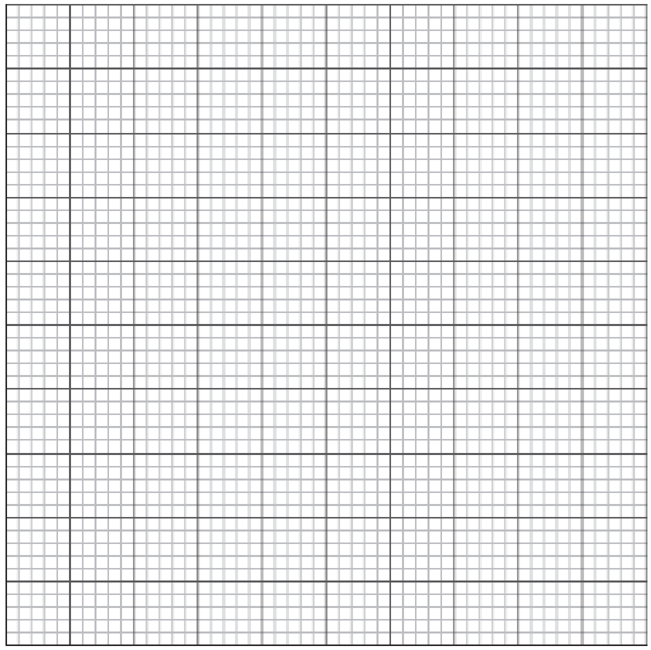
[9 marks]

Total [16 marks]

Examiners report

Many students could not sketch the function. There was confusion between the vertical and horizontal line test for one-to-one functions. A significant number of students gave long and inaccurate explanations for a one-to-one function. Finding the inverse was done very well by most students although the notation used was generally poor. The domain of the inverse was ignored by many or done incorrectly even if the sketch was correct. Many did not make the connections between the parts of the question. An example of this was the number of students who spent time finding the point of intersection in part e) even though it was given in d).

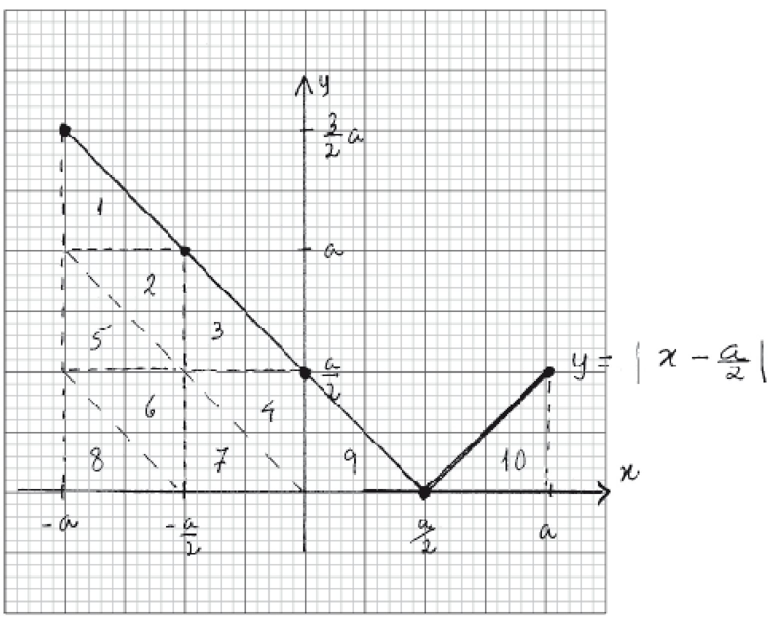
(a) Let $a > 0$. Draw the graph of $y = \left|x - \frac{a}{2}\right|$ for $-a \leqslant x \leqslant a$ on the grid below.



(b) Find k such that $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = k \int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x$.

Markscheme

(a)



A1A1

Note: Award ***A1*** for the correct x-intercept,

A1 for completely correct graph.

(b) **METHOD 1**

the area under the graph of $y = \left|x - \frac{a}{2}\right|$ for $-a \leqslant x \leqslant a$, can be divided into ten congruent triangles; ***M1A1***

the area of eight of these triangles is given by $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x$ and the areas of the other two by $\int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x$ ***M1A1***

so, $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = 4 \int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x \Rightarrow k = 4$ ***A1*** ***N0***

METHOD 2

use area of trapezium to calculate ***M1***

$\int_{-a}^{0} \left| x - \frac{a}{2} \right| \mathrm{d}x = a \times \frac{1}{2}\left(\frac{3a}{2} + \frac{a}{2}\right) = a^2$ ***A1***

and area of two triangles to obtain ***M1***

$\int_{0}^{a} \left| x - \frac{a}{2} \right| \mathrm{d}x = 2 \times \frac{1}{2}\left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$ ***A1***

so, $k = 4$ ***A1*** ***N0***

METHOD 3

use integration to find the area under the curve

$\int_{-a}^{0} \left| x - \frac{a}{2} \right| \mathrm{d}x = \int_{-a}^{0} -x + \frac{a}{2}\mathrm{d}x$ ***M1***

$= \left[-\frac{x^2}{2} + \frac{a}{2}x\right]_{-a}^{0} = \frac{a^2}{2} + \frac{a^2}{2} = a^2$ ***A1***

and

$\int_{0}^{a} \left| x - \frac{a}{2} \right| \mathrm{d}x = \int_{0}^{\frac{a}{2}} -x + \frac{a}{2}\mathrm{d}x + \int_{\frac{a}{2}}^{a} x - \frac{a}{2}\mathrm{d}x$ ***M1***

$= \left[-\frac{x^2}{2} + \frac{a}{2}x\right]_{0}^{\frac{a}{2}} + \left[\frac{x^2}{2} - \frac{a}{2}x\right]_{\frac{a}{2}}^{a} = \frac{a^2}{8} + \frac{a^2}{4} + \frac{a^2}{2} - \frac{a^2}{2} - \frac{a^2}{8} + \frac{a^2}{4} = \frac{a^2}{4}$ ***A1***

so, $k = 4$ ***A1*** ***N0***

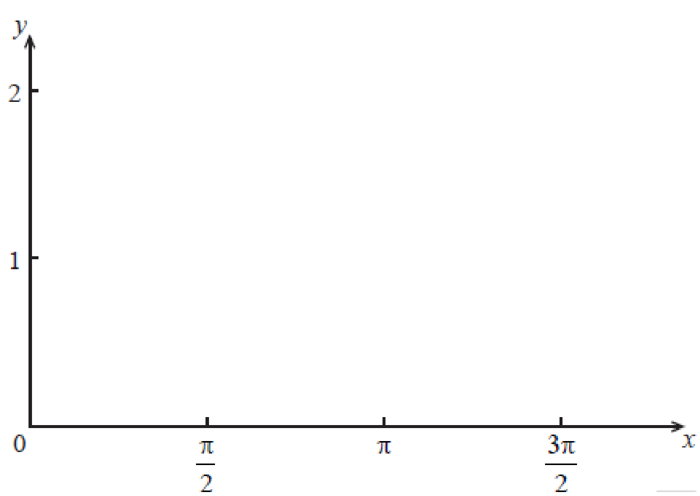
[7 marks]

Examiners report

Most candidates attempted this question but very often produced sketches lacking labels on axes and intercepts or ignored the domain of the function. For part (b) many candidates attempted to use integration to find the areas but seldom considered the absolute value. A small number of candidates used geometrical methods to determine the areas, showing good understanding of the problem.

Given that $f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

a. sketch the graph of f; [1]

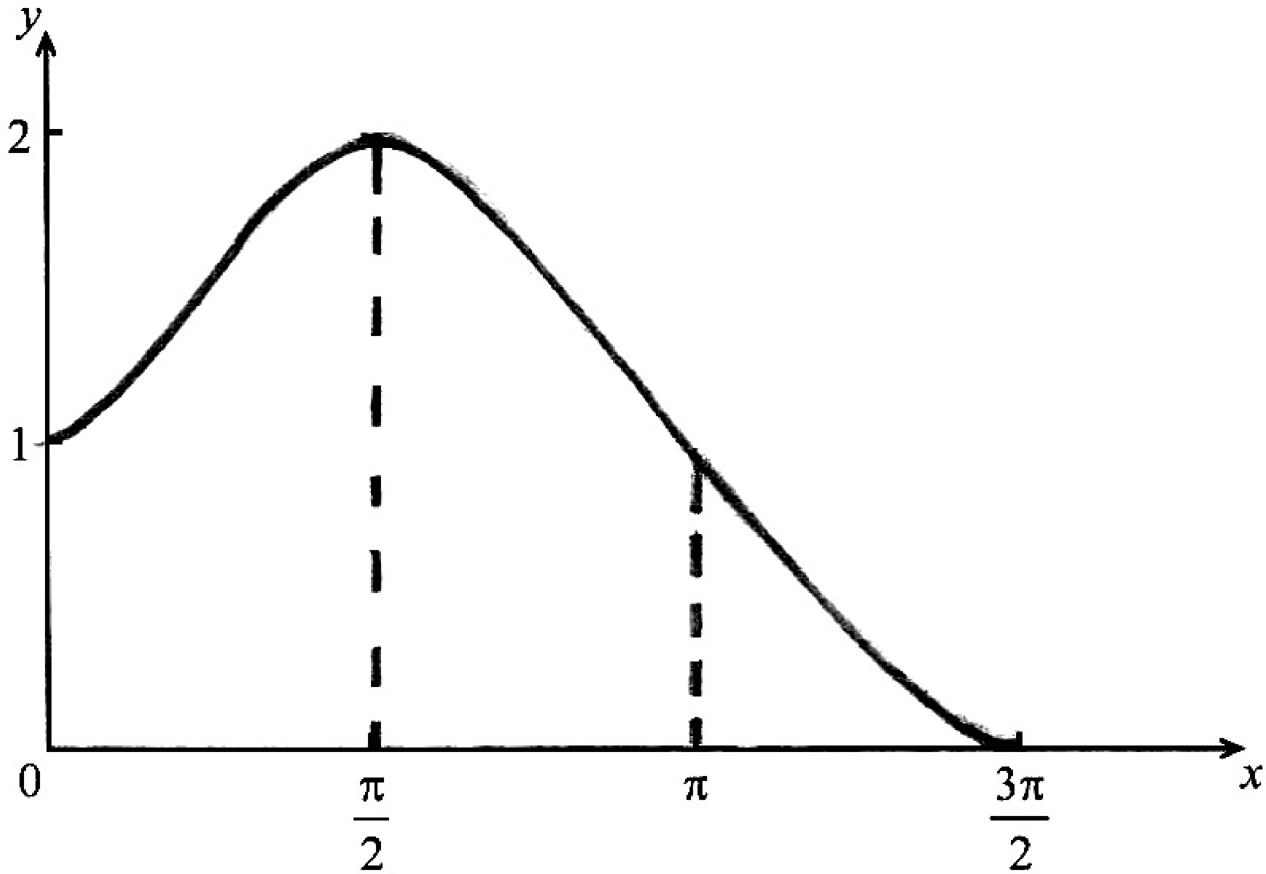


b. show that $(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$; [1]

c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. [4]

Markscheme

a.



A1

[1 mark]

b. $(1+\sin x)^2 = 1 + 2\sin x + \sin^2 x$

$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$ ***A1***

$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$ ***AG***

[1 mark]

c. $V = \pi \int_0^{\frac{3\pi}{2}} (1+\sin x)^2 \mathrm{d}x$ ***(M1)***

$= \pi \int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right) \mathrm{d}x$

$= \pi\left[\frac{3}{2}x - 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{3\pi}{2}}$ ***A1***

$= \frac{9\pi^2}{4} + 2\pi$ ***A1A1***

[4 marks]

Examiners report

a. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

b. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

c. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < \mathrm{e}^2$.

a. (i) Solve the equation $f'(x) = 0$. [5]

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f .

b. Show that there is a point of inflexion on the graph and determine its coordinates. [5]

c. Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. [3]

d. Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < x < \mathrm{e}^2$. [6]

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g .

(iii) Find the values of x such that $h(x) > g(x)$.

Markscheme

a. (i) $f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \frac{1 - \ln x}{x^2}$

so $f'(x) = 0$ when $\ln x = 1$, *i.e.* $x = \mathrm{e}$ ***A1***

(ii) $f'(x) > 0$ when $x < \mathrm{e}$ and $f'(x) < 0$ when $x > \mathrm{e}$ ***R1***

hence local maximum ***AG***

Note: Accept argument using correct second derivative.

(iii) $y \leqslant \frac{1}{\mathrm{e}}$ ***A1***

[5 marks]

b. $f''(x) = \frac{x^2\frac{-1}{x} - (1 - \ln x)2x}{x^4}$ ***M1***

$= \frac{-x - 2x + 2x\ln x}{x^4}$

$= \frac{-3 + 2\ln x}{x^3}$ ***A1***

Note: May be seen in part (a).

$f''(x) = 0$ ***(M1)***

$-3 + 2\ln x = 0$

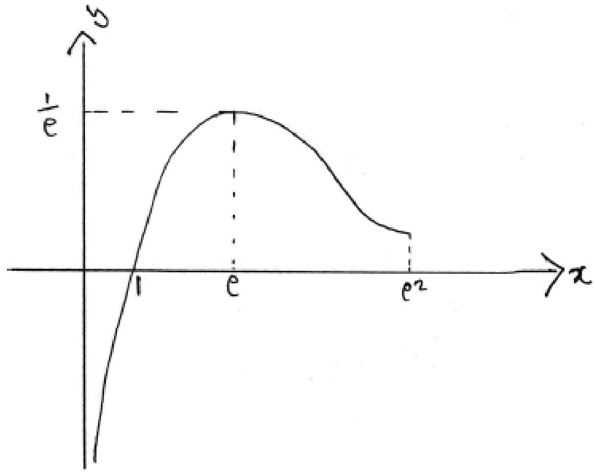
$x = \mathrm{e}^{\frac{3}{2}}$

since $f''(x) < 0$ when $x < \mathrm{e}^{\frac{3}{2}}$ and $f''(x) > 0$ when $x > \mathrm{e}^{\frac{3}{2}}$ ***R1***

then point of inflexion $\left(\mathrm{e}^{\frac{3}{2}}, \frac{3}{2\mathrm{e}^{\frac{3}{2}}}\right)$ ***A1***

[5 marks]

c.

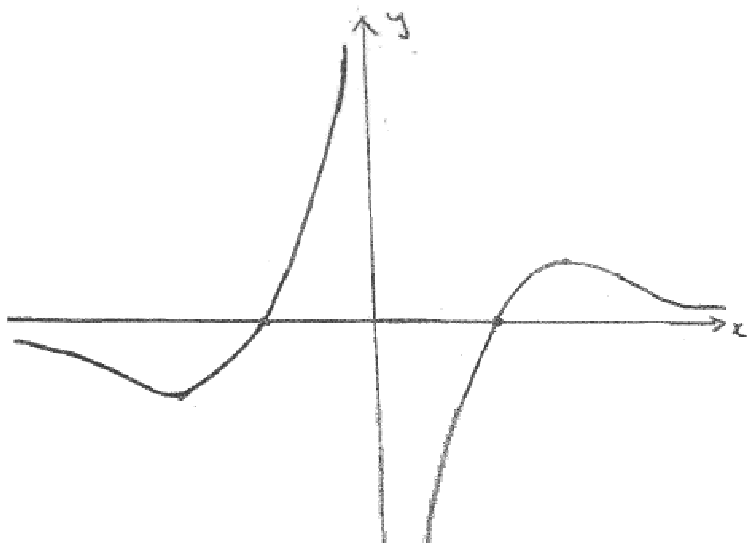


A1A1A1

Note: Award ***A1*** for the maximum and intercept, ***A1*** for a vertical asymptote and ***A1*** for shape (including turning concave up).

[3 marks]

d. (i)

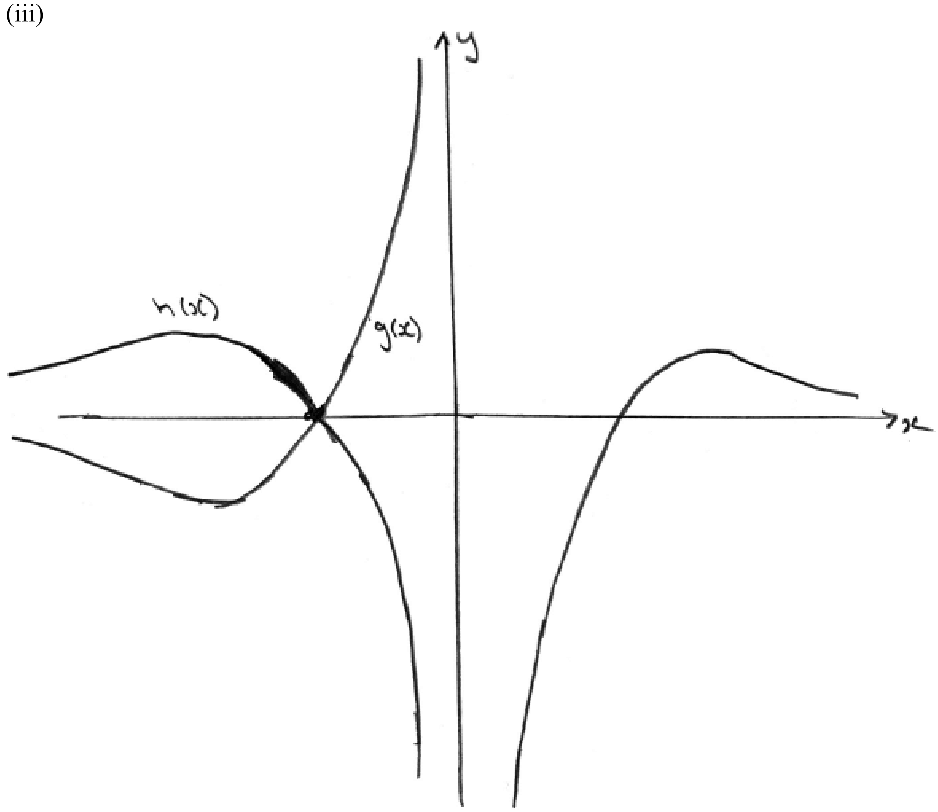


A1A1

Note: Award ***A1*** for each correct branch.

(ii) all real values ***A1***

(iii)



(M1)(A1)

Note: Award ***(M1)(A1)*** for sketching the graph of h, ignoring any graph of g.

$-\mathrm{e}^2 < x < -1$ (accept $x < -1$) ***A1***

[6 marks]

Examiners report

a. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

b. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

c. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

d. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

A tangent to the graph of $y = \ln x$ passes through the origin.

(a) Sketch the graphs of $y = \ln x$ and the tangent on the same set of axes, and hence find the equation of the tangent.

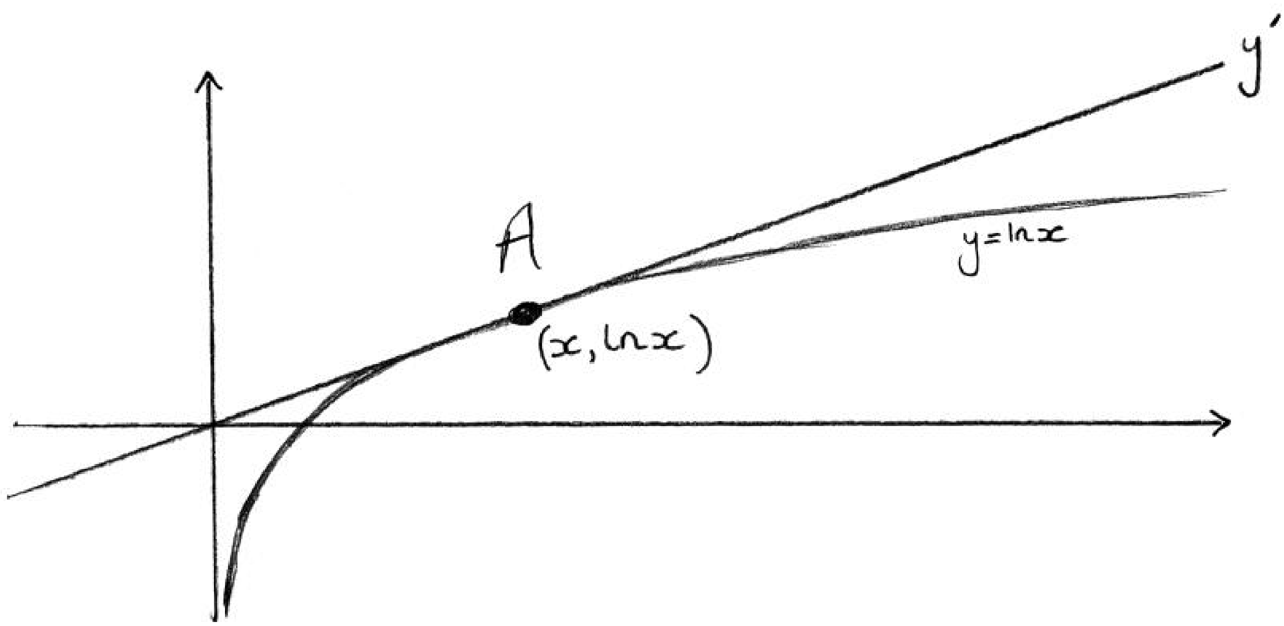
(b) Use your sketch to explain why $\ln x \leqslant \frac{x}{\mathrm{e}}$ for $x > 0$.

(c) Show that $x^{\mathrm{e}} \leqslant \mathrm{e}^{x}$ for $x > 0$.

(d) Determine which is larger, π^{e} or e^{π} .

Markscheme

(a)



A3

Note: Award ***A1*** for each graph

A1 for the point of tangency.

point on curve and line is $(a,\ \ln a)$ ***(M1)***

$y = \ln(x)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}$ (when $x = a$) ***(M1)A1***

EITHER

gradient of line, m, through (0, 0) and $(a,\ \ln a)$ is $\frac{\ln a}{a}$ ***(M1)A1***

$\Rightarrow \frac{\ln a}{a} = \frac{1}{a} \Rightarrow \ln a = 1 \Rightarrow a = \mathrm{e} \Rightarrow m = \frac{1}{\mathrm{e}}$ ***M1A1***

OR

$y - \ln a = \frac{1}{a}(x - a)$ ***(M1)A1***

passes through 0 if

$\ln a - 1 = 0$ ***M1***

$a = e \Rightarrow m = \frac{1}{\mathrm{e}}$ ***A1***

THEN

$\therefore y = \frac{1}{\mathrm{e}}x$ ***A1***

[11 marks]

(b) the graph of $\ln x$ never goes above the graph of $y = \frac{1}{\mathrm{e}}x$, hence $\ln x \leqslant \frac{x}{\mathrm{e}}$ ***R1***

[1 mark]

(c) $\ln x \leqslant \frac{x}{\mathrm{e}} \Rightarrow \mathrm{e} \ln x \leqslant x \Rightarrow \ln x^{\mathrm{e}} \leqslant x$ ***M1A1***

exponentiate both sides of $\ln x^{\mathrm{e}} \leqslant x \Rightarrow x^{\mathrm{e}} \leqslant \mathrm{e}^{x}$ ***R1AG***

[3 marks]

(d) equality holds when $x = \mathrm{e}$ ***R1***

letting $x = \pi \Rightarrow \pi^{\mathrm{e}} < \mathrm{e}^{\pi}$ ***A1*** ***N0***

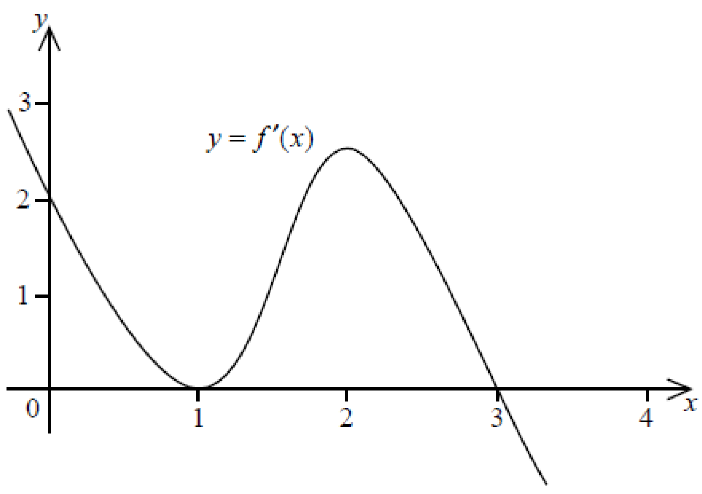
[2 marks]

Total [17 marks]

Examiners report

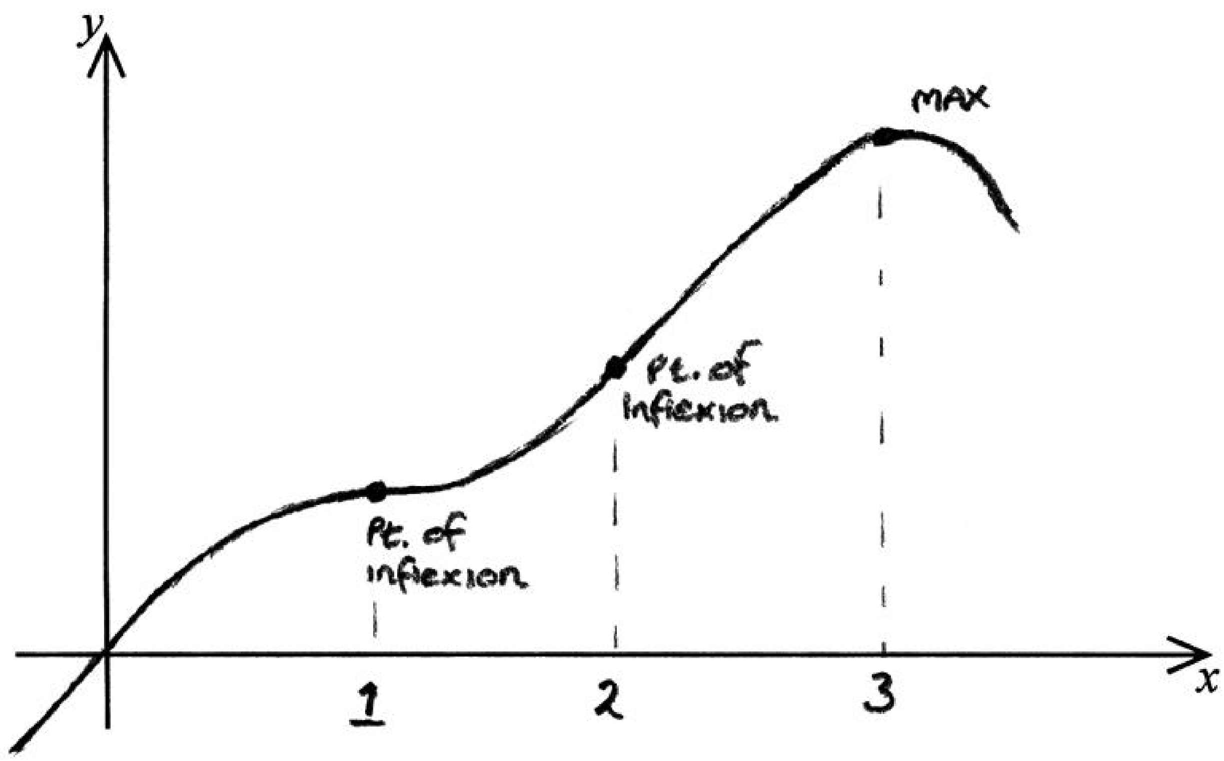
This was the least accessible question in the entire paper, with very few candidates achieving high marks. Sketches were generally done poorly, and candidates failed to label the point of intersection. A ‘dummy’ variable was seldom used in part (a), hence in most cases it was not possible to get more than 3 marks. There was a lot of good guesswork as to the coordinates of the point of intersection, but no reasoning showed. Many candidates started with the conclusion in part (c). In part (d) most candidates did not distinguish between the inequality and strict inequality.

The diagram below shows a sketch of the gradient function $f'(x)$ of the curve $f(x)$.



On the graph below, sketch the curve $y = f(x)$ given that $f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.

Markscheme



A5

Note: Award ***A1*** for origin

A1 for shape

A1 for maximum

A1 for each point of inflexion.

[5 marks]

Examiners report

A reasonable number of candidates answered this correctly, although some omitted the 2^{nd} point of inflection.

Consider $f(x) = \frac{x^2-5x+4}{x^2+5x+4}$.

(a) Find the equations of all asymptotes of the graph of *f*.

(b) Find the coordinates of the points where the graph of *f* meets the *x* and *y* axes.

(c) Find the coordinates of

(i) the maximum point and justify your answer;

(ii) the minimum point and justify your answer.

(d) Sketch the graph of *f*, clearly showing all the features found above.

(e) **Hence**, write down the number of points of inflexion of the graph of *f*.

Markscheme

(a) $x^2 + 5x + 4 = 0 \Rightarrow x = -1$ or $x = -4$ ***(M1)***

so vertical asymptotes are $x = -1$ and $x = -4$ ***A1***

as $x \to \infty$ then $y \to 1$ so horizontal asymptote is $y = 1$ ***(M1)A1***

[4 marks]

(b) $x^2 - 5x + 4 = 0 \Rightarrow x = 1$ or $x = 4$ ***A1***

$x = 0 \Rightarrow y = 1$ ***A1***

so intercepts are (1, 0), (4, 0) and (0,1)

[2 marks]

(c) (i) $f'(x) = \frac{(x^2+5x+4)(2x-5)-(x^2-5x+4)(2x+5)}{(x^2+5x+4)^2}$ ***M1A1A1***

$= \frac{10x^2-40}{(x^2+5x+4)^2} \quad \left(= \frac{10(x-2)(x+2)}{(x^2+5x+4)^2}\right)$ ***A1***

$f'(x) = 0 \Rightarrow x = \pm 2$ ***M1***

so the points under consideration are (–2, –9) and $\left(2, -\frac{1}{9}\right)$ ***A1A1***

looking at the sign either side of the points (or attempt to find $f''(x)$) ***M1***

e.g. if $x = -2^-$ then $(x-2)(x+2) > 0$ and if $x = -2^+$ then $(x-2)(x+2) < 0$,

therefore (–2, –9) is a maximum ***A1***

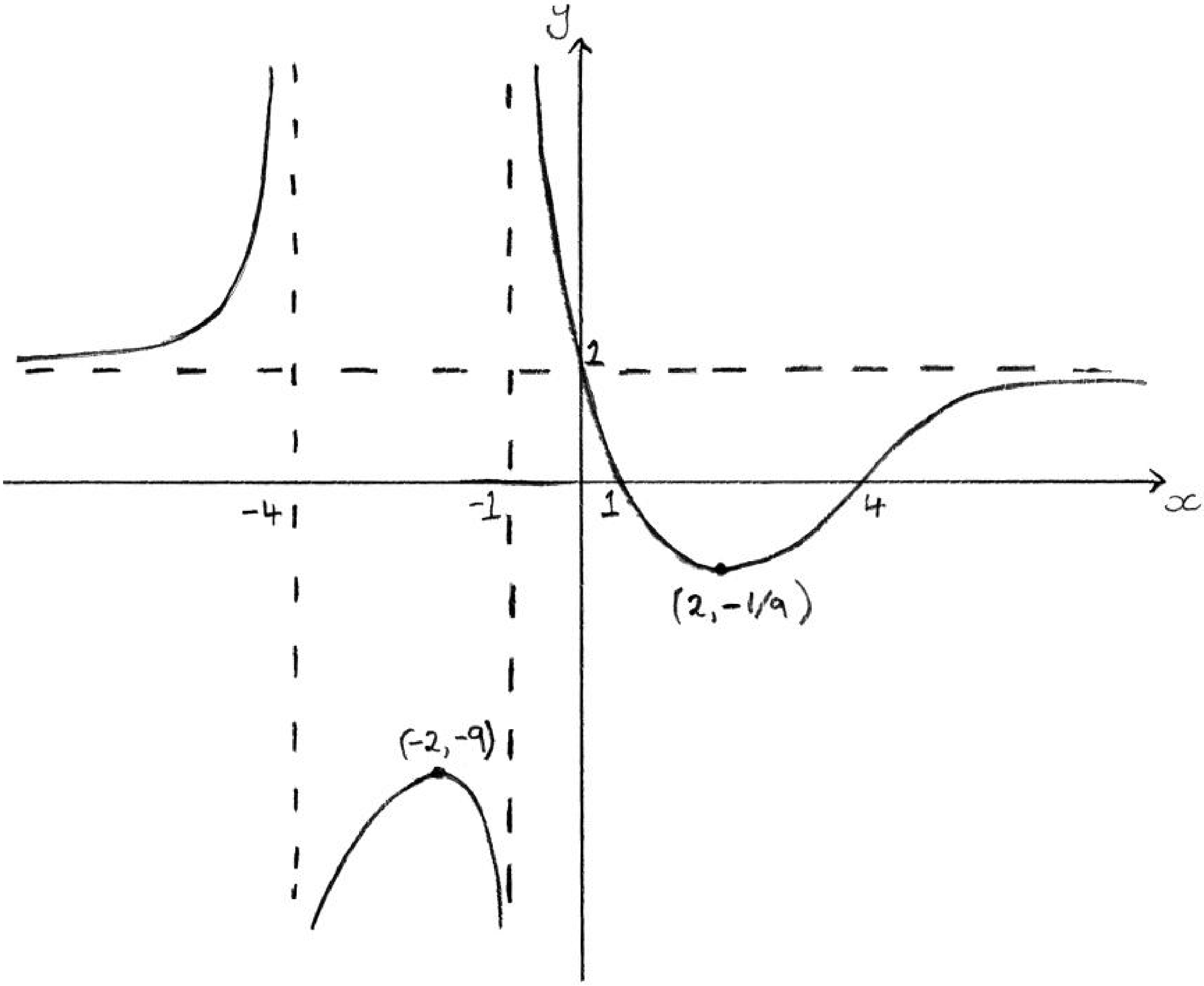
(ii) *e.g.* if $x = 2^-$ then $(x-2)(x+2) < 0$ and if $x = 2^+$ then $(x-2)(x+2) > 0$,

therefore $\left(2, -\frac{1}{9}\right)$ is a minimum ***A1***

Note: Candidates may find the minimum first.

[10 marks]

(d)



A3

Note: Award *A1* for each branch consistent with and including the features found in previous parts.

[3 marks]

(e) one ***A1***

[1 mark]

Total [20 marks]

Examiners report

This was the most successfully answered question in part B, in particular parts (a), (b) and (c). In part (a) the horizontal asymptote was often missing (or $x = 4$, $x = 1$ given). Part (b) was well done. Use of the quotient rule was well done in part (c) and many simplified correctly. There was knowledge of max/min and how to justify their answer, usually with a sign diagram but also with the second derivative. A common misconception was that, as $-9 < -\frac{1}{9}$, the minimum is at (–2, –9). In part (d) many candidates were unable to sketch the graph consistent with the main features that they had determined before. Very few candidates answered part (e) correctly.

Let *f* be a function defined by $f(x) = x - \arctan x$, $x \in \mathbb{R}$.

(a) Find $f(1)$ and $f\left(-\sqrt{3}\right)$.

(b) Show that $f(-x) = -f(x)$, for $x \in \mathbb{R}$.

(c) Show that $x - \frac{\pi}{2} < f(x) + \frac{\pi}{2}$, for $x \in \mathbb{R}$.

(d) Find expressions for $f'(x)$ and $f''(x)$. Hence describe the behaviour of the graph of *f* at the origin and justify your answer.

(e) Sketch a graph of *f* , showing clearly the asymptotes.

(f) Justify that the inverse of *f* is defined for all $x \in \mathbb{R}$ and sketch its graph.

Markscheme

(a) $f(1) = 1 - \arctan 1 = 1 - \frac{\pi}{4}$ ***A1***

$f(-\sqrt{3}) = -\sqrt{3} - \arctan(-\sqrt{3}) = -\sqrt{3} + \frac{\pi}{3}$ ***A1***

[2 marks]

(b) $f(-x) = -x - \arctan(-x)$ ***M1***

$= -x + \arctan x$ ***A1***

$= -(x - \arctan x)$

$= -f(x)$ ***AG*** ***N0***

[2 marks]

(c) as $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$ ***A1***

$\Rightarrow -\frac{\pi}{2} < -\arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$

then by adding *x* (or equivalent) ***R1***

we have $x - \frac{\pi}{2} < x - \arctan x < x + \frac{\pi}{2}$ ***AG*** ***N0***

[2 marks]

(d) $f'(x) = 1 - \frac{1}{1+x^2}$ or $\frac{x^2}{1+x^2}$ ***A1A1***

$f''(x) = \frac{2x(1+x^2)-2x^3}{(1+x^2)^2}$ or $\frac{2x}{(1+x^2)^2}$ ***M1A1***

$f'(0) = f''(0) = 0$ ***A1A1***

EITHER

as $f'(x) \geqslant 0$ for all values of $x \in \mathbb{R}$

($(0,\ 0)$is not an extreme of the graph of *f* (or equivalent)) ***R1***

OR

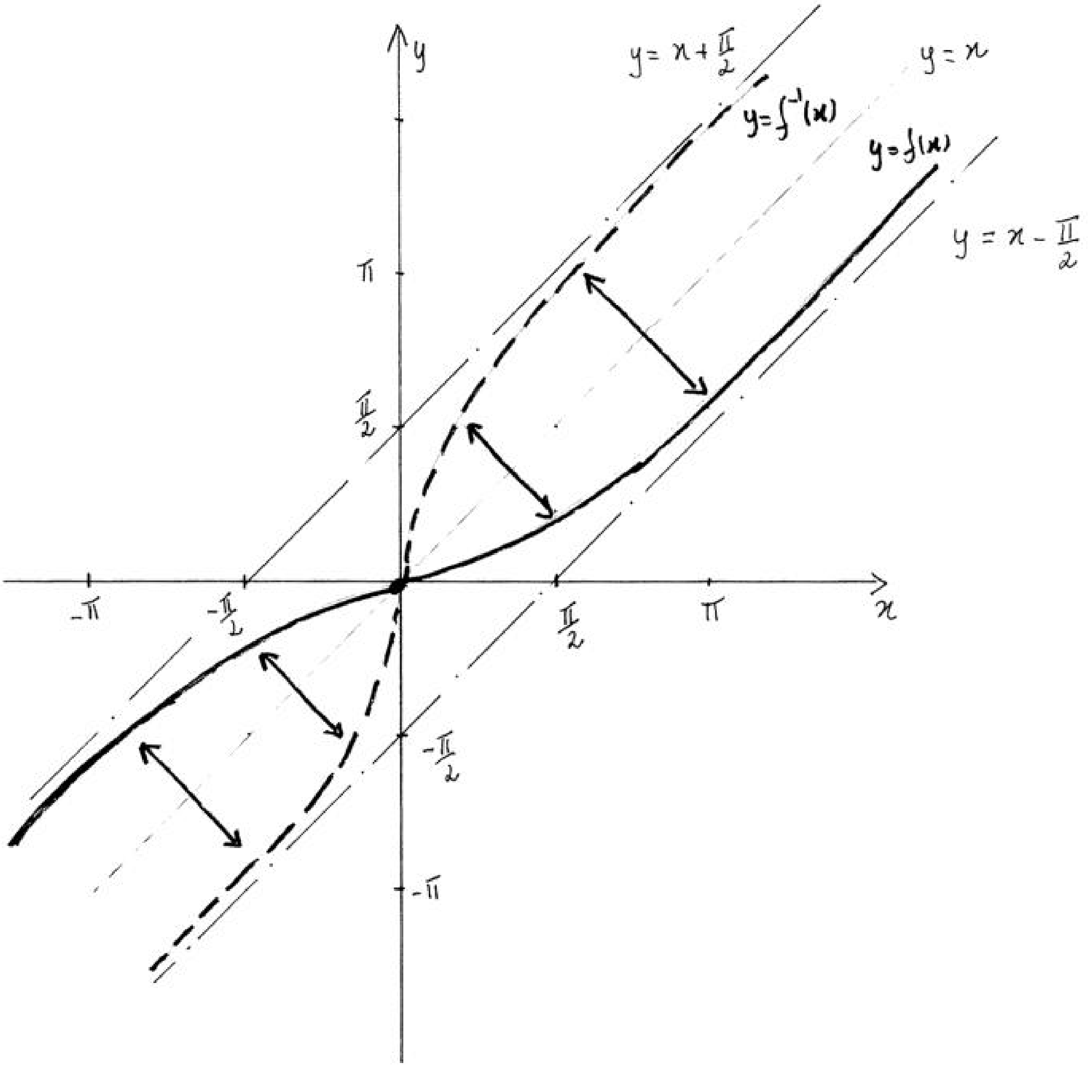
as $f''(x) > 0$for positive values of *x* and $f''(x) < 0$ for negative values of *x* ***R1***

THEN

(0, 0) is a point of inflexion of the graph of *f* (with zero gradient) ***A1*** ***N2***

[8 marks]

(e)



A1A1A1

Note: Award ***A1*** for both asymptotes.

A1 for correct shape (concavities) $x < 0$.

A1 for correct shape (concavities) $x > 0$.

[3 marks]

(f) (see sketch above)

as *f* is increasing (and therefore one-to-one) and its range is $\mathbb{R}$,

f^{-1} is defined for all $x \in \mathbb{R}$ ***R1***

use the result that the graph of $y = f^{-1}(x)$ is the reflection

in the line *y* = *x* of the graph of $y = f(x)$ to draw the graph of f^{-1} ***(M1)A1***

[3 marks]

Total [20 marks]

Examiners report

Parts of this question were answered quite well by many candidates. A few candidates had difficulties with domain of arctan in part (a) and in justifying their reasoning in parts (b) and (c). In part (d) although most candidates were successful in finding the expressions of the derivatives and their values at *x* = 0, many were unable to use the results to find the nature of the curve at the origin. Very few candidates were successful in answering parts (e) and (f).

---

a. (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$. [9]

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

b. Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. [8]

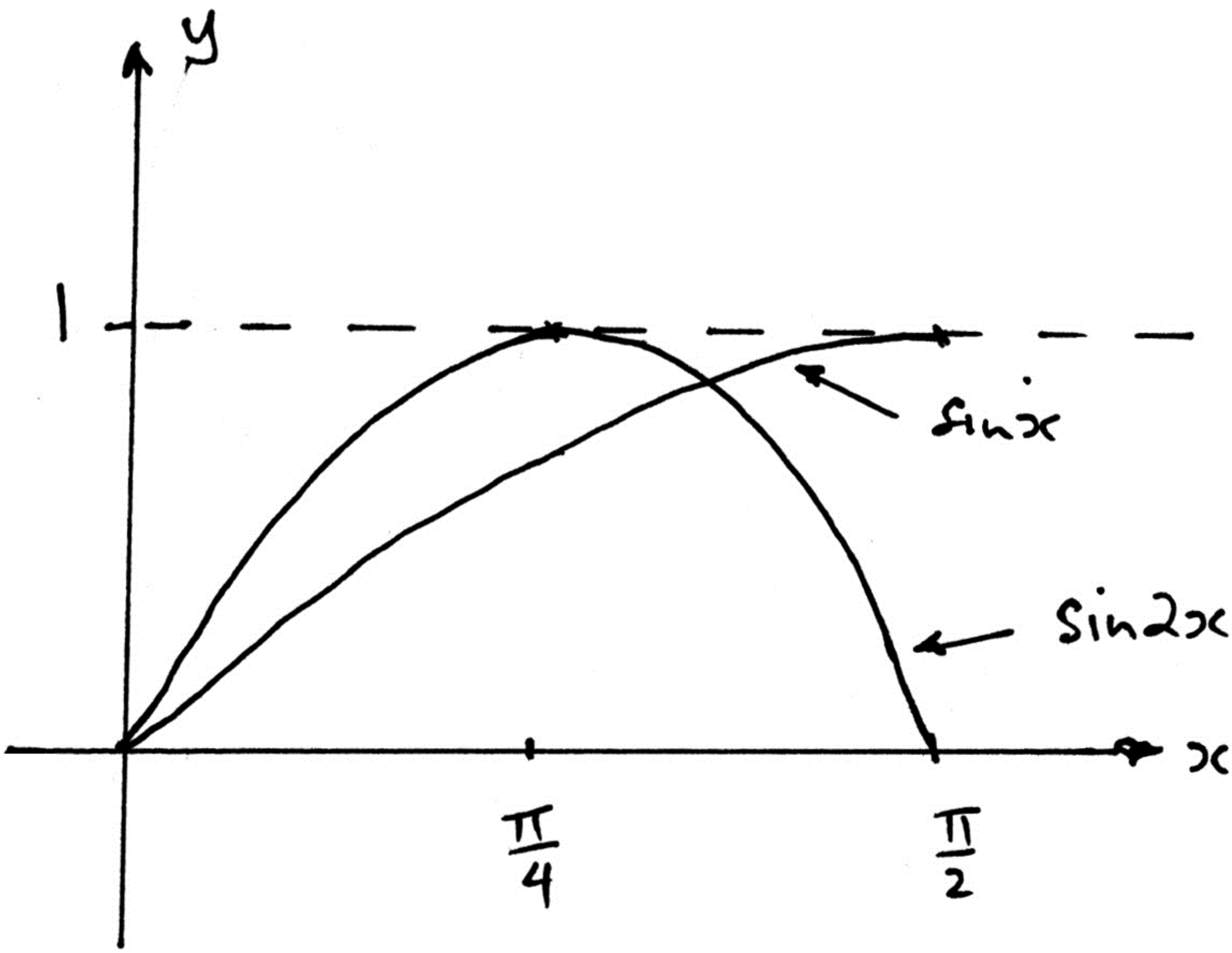
c. The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$. [8]

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

Markscheme

a. (i)



A2

Note: Award ***A1*** for correct $\sin x$, ***A1*** for correct $\sin 2x$.

Note: Award ***A1A0*** for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x$, $0 \leqslant x \leqslant \frac{\pi}{2}$
$2\sin x\cos x - \sin x = 0$ ***M1***
$\sin x(2\cos x - 1) = 0$
$x = 0, \frac{\pi}{3}$ ***A1A1*** ***N1N1***

(iii) area $= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\mathrm{d}x$ ***M1***
Note: Award ***M1*** for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

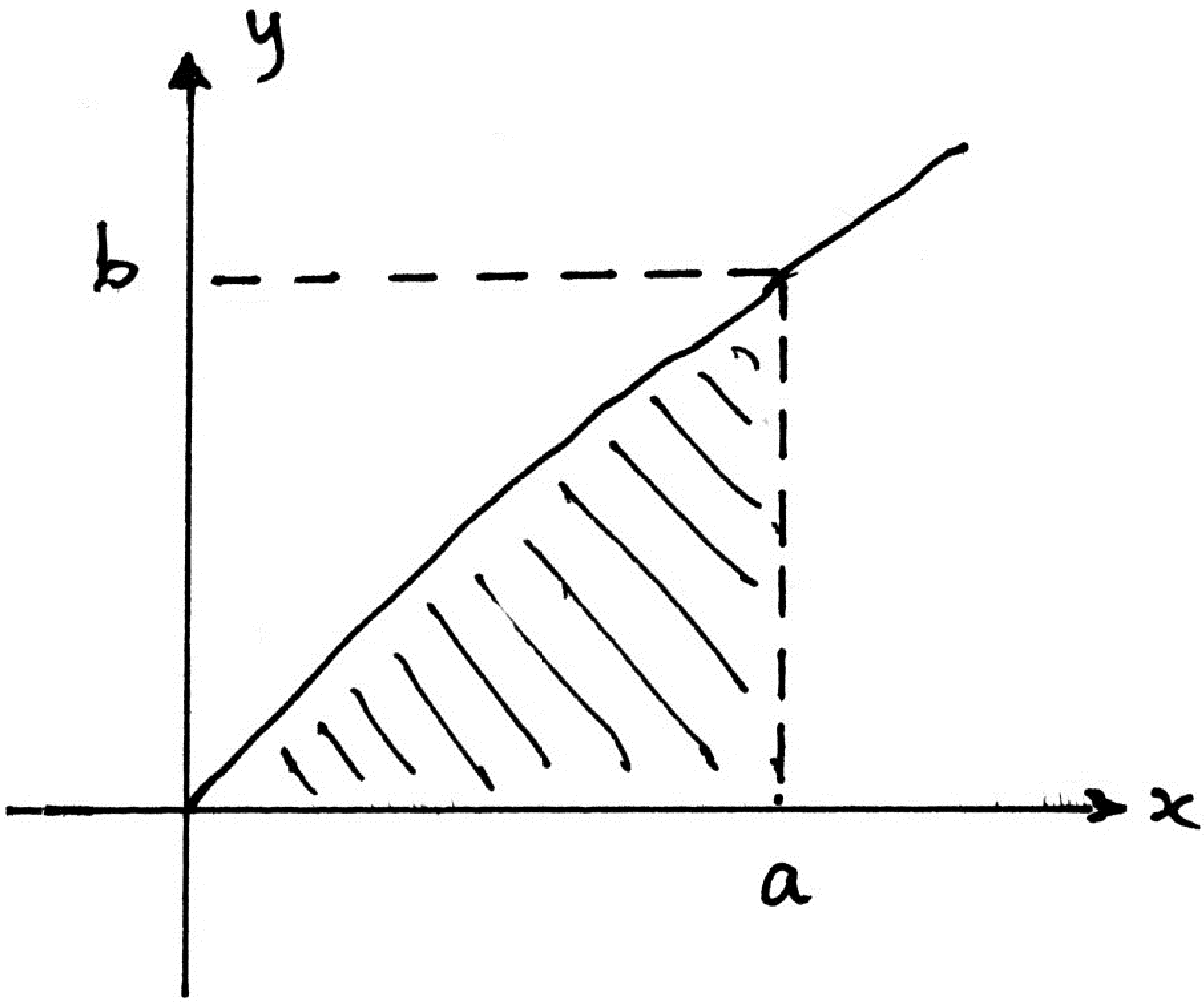
$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ ***A1***
$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ ***(M1)***
$= \frac{3}{4} - \frac{1}{2}$
$= \frac{1}{4}$ ***A1***
[9 marks]

b. $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\mathrm{d}\theta$ ***M1A1A1***

Note: Award ***M1*** for substitution and reasonable attempt at finding expression for dx in terms of dθ , first ***A1*** for correct limits, second ***A1*** for correct substitution for dx .

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\mathrm{d}\theta$ ***A1***
$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\mathrm{d}\theta$ ***M1***
$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ ***A1***
$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ ***(M1)***
$= \frac{2\pi}{3} - \sqrt{3}$ ***A1***
[8 marks]

c. (i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$ ***R1***

$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$ ***AG***

(ii) $f(x) = \arcsin \frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ ***A1***

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\mathrm{d}x$ ***M1A1A1***

Note: Award ***A1*** for the limit $\frac{\pi}{6}$ seen anywhere, ***A1*** for all else correct.

$= \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$ ***A1***

$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ ***A1***

Note: Award no marks for methods using integration by parts.

[8 marks]

Examiners report

a. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution.
The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the dx expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required.
Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

b. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
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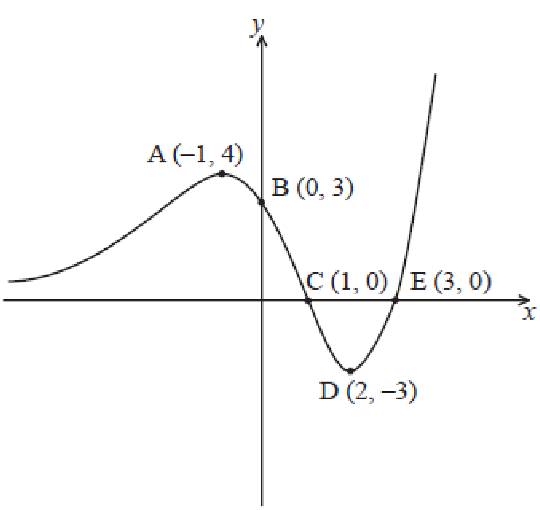
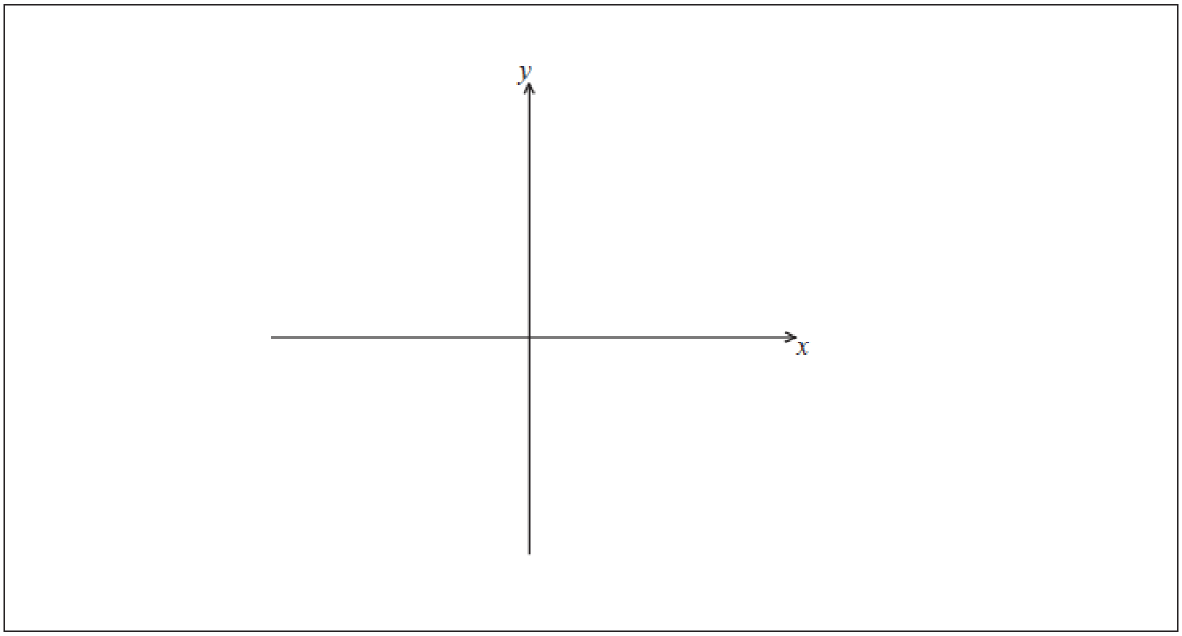
Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

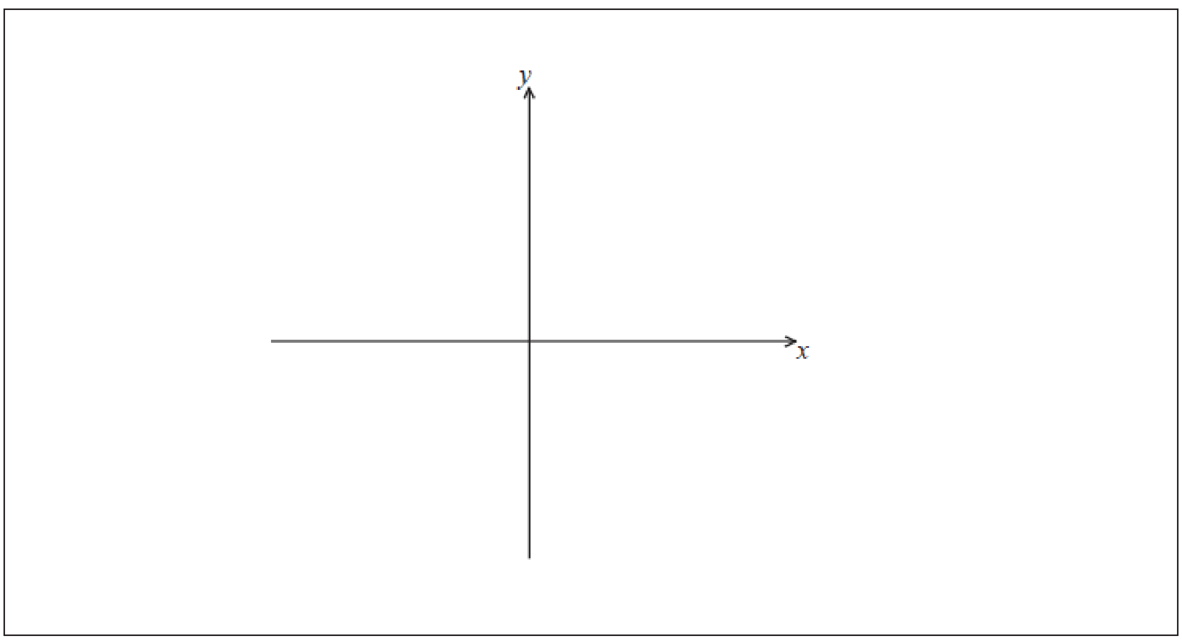
The graph of $y = f(x)$ is shown below, where A is a local maximum point and D is a local minimum point.



a. On the axes below, sketch the graph of $y = \frac{1}{f(x)}$, clearly showing the coordinates of the images of the points A, B and D, labelling them A$'$, B$'$, and D$'$ respectively, and the equations of any vertical asymptotes. [3]

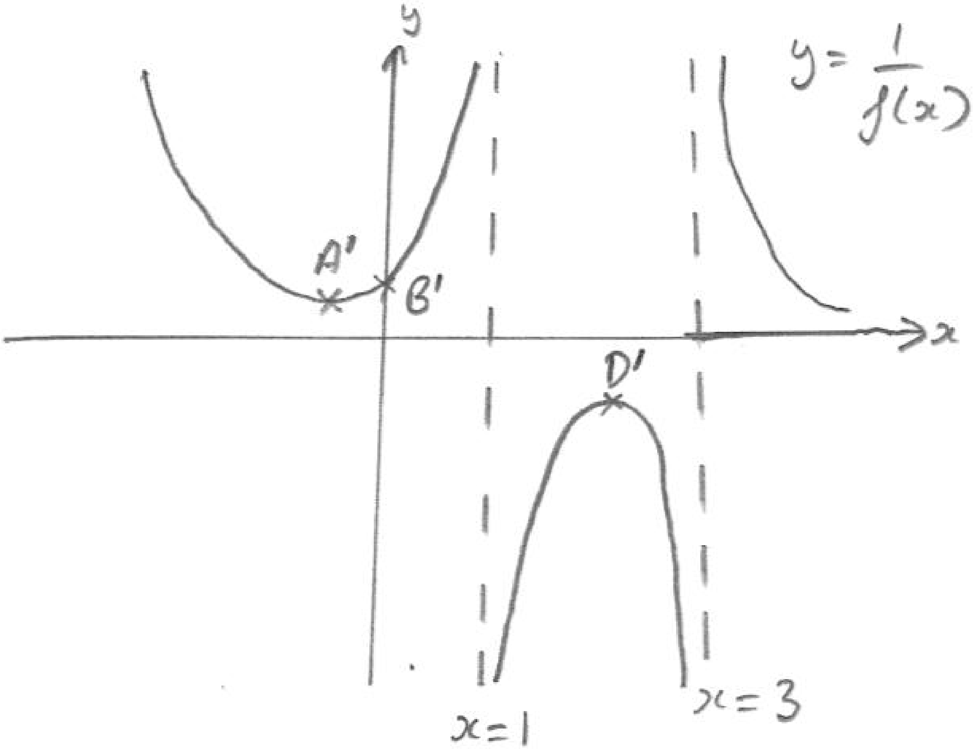


b. On the axes below, sketch the graph of the derivative $y = f'(x)$, clearly showing the coordinates of the images of the points A and D, labelling them A$''$ and D$''$ respectively. [3]



Markscheme

a.

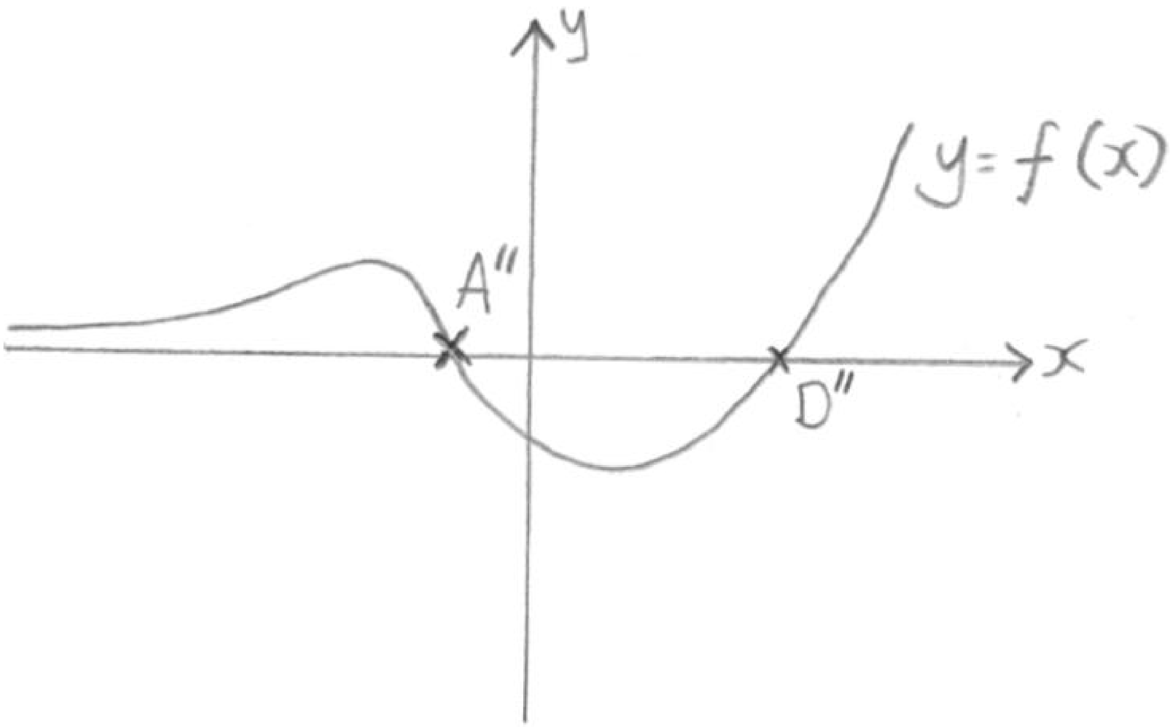


A1A1A1

Note: Award ***A1*** for correct shape.
Award ***A1*** for two correct asymptotes, and $x = 1$ and $x = 3$.
Award ***A1*** for correct coordinates, $\mathrm{A}'\left(-1, \frac{1}{4}\right)$, $\mathrm{B}'\left(0, \frac{1}{3}\right)$ and $\mathrm{D}'\left(2, -\frac{1}{3}\right)$.

[3 marks]

b.



A1A1A1

Note: Award ***A1*** for correct general shape including the horizontal asymptote.
Award ***A1*** for recognition of 1 maximum point and 1 minimum point.
Award ***A1*** for correct coordinates, $\mathrm{A}''(-1, 0)$ and $\mathrm{D}''(2, 0)$.

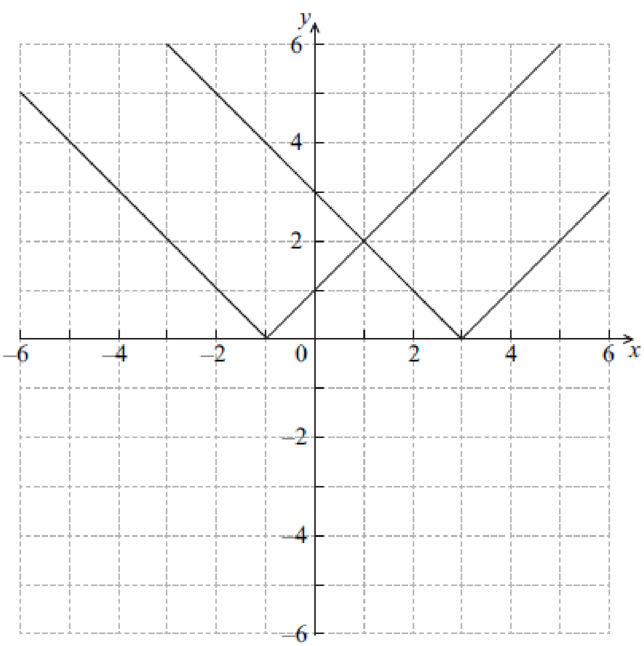
[3 marks]

Examiners report

a. Solutions to this question were generally disappointing. In (a), the shape of the graph was often incorrect and many candidates failed to give the equations of the asymptotes and the coordinates of the image points. In (b), many candidates produced incorrect graphs although the coordinates of the image points were often given correctly.

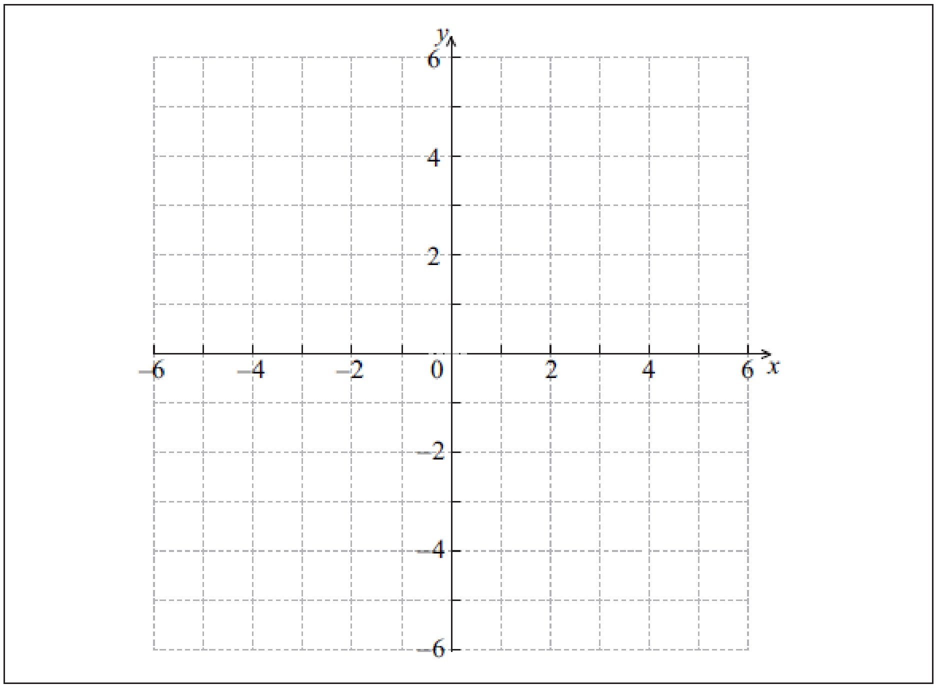
b. Solutions to this question were generally disappointing. In (a), the shape of the graph was often incorrect and many candidates failed to give the equations of the asymptotes and the coordinates of the image points. In (b), many candidates produced incorrect graphs although the coordinates of the image points were often given correctly.

The graphs of $y = |x + 1|$ and $y = |x - 3|$ are shown below.



Let $f(x) = |x + 1| - |x - 3|$.

a. Draw the graph of $y = f(x)$ on the blank grid below. [4]



b. Hence state the value of [4]

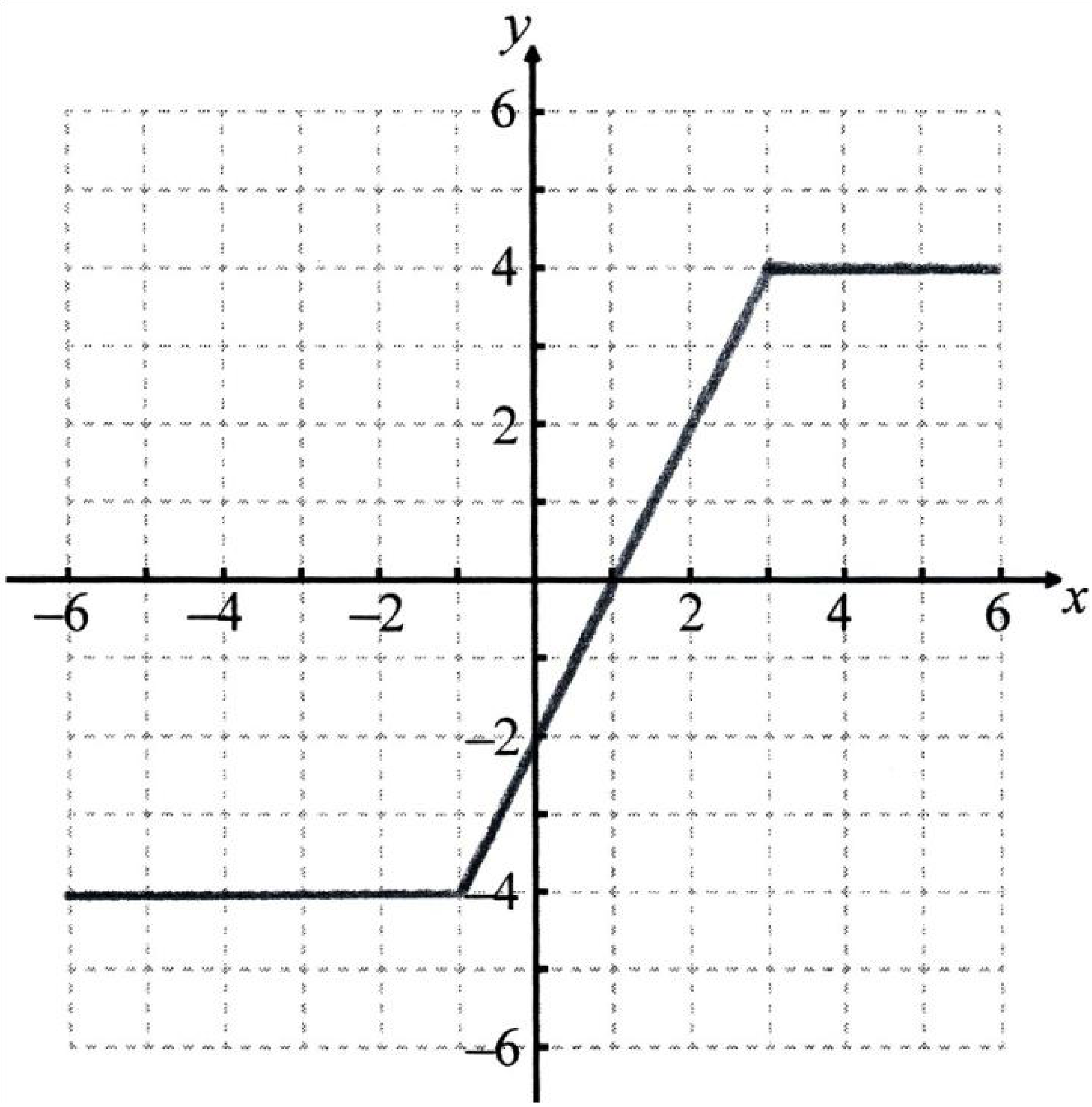
(i) $f'(-3)$;

(ii) $f'(2.7)$;

(iii) $\int_{-3}^{-2} f(x)dx$.

Markscheme

a.



M1A1A1A1

Note: Award ***M1*** for any of the three sections completely correct, ***A1*** for each correct segment of the graph. ***[4 marks]***

b. (i) 0 ***A1***

(ii) 2 ***A1***

(iii) finding area of rectangle ***(M1)***

-4 ***A1***

Note: Award ***M1A0*** for the answer 4.

[4 marks]

Examiners report

a. Most candidates were able to produce a good graph, and many were able to interpret that to get correct answers to part (b). The most common error was to give 4 as the answer to (b) (iii). Some candidates did not recognise that the "hence" in the question meant that they had to use their graph to obtain their answers to part (b).

b. Most candidates were able to produce a good graph, and many were able to interpret that to get correct answers to part (b). The most common error was to give 4 as the answer to (b) (iii). Some candidates did not recognise that the "hence" in the question meant that they had to use their graph to obtain their answers to part (b).